# Problem Set 4

1. **Fibonacci Fun:** Prove the nth Fibonacci number is equal to $\frac{\phi^n-(1-\phi)^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio, and satisfies the identity $\phi^2 = \phi + 1$. Be sure to state what style of proof you are using.

2. Hard examples for stable marriage algorithm.

   In the last homework, you ran the traditional propose and reject algorithm on the following instance:

   Men's preference list:

| | | | | |
|---|---|---|---|---|
| 1 | $A$ | $B$ | $C$ | $D$ |
| 2 | $B$ | $C$ | $A$ | $D$ |
| 3 | $C$ | $A$ | $B$ | $D$ |
| 4 | $A$ | $B$ | $C$ | $D$ |

   Women's preference list:

| | | | | |
|---|---|---|---|---|
| $A$ | 2 | 3 | 4 | 1 |
| $B$ | 3 | 4 | 1 | 2 |
| $C$ | 4 | 1 | 2 | 3 |
| $D$ | 1 | 2 | 3 | 4 |

   (a) In class we showed that the propose and reject algorithm must terminate after at most $n^2$ proposals. Prove a sharper bound showing that the algorithm must terminate after at most $n(n-1)+1$ proposals. Conclude that the above example is a worst case instance for $n = 4$. How many days does the algorithm take on this instance?

   (b) **Extra credit:** generalize the above example to arbitraty $n$ and prove rigorously that the algorithm makes $n(n-1)+1$ proposals on your example. How many days does the algorithm take?

3. **Man-optimal, Woman-optimal** In an particular instance of the stable marriage problem with $n$ men and $n$ women, it turns out that there are exactly three distinct stable matchings, $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$. Also, each woman $W$ has a different partner in the three matchings. Therefore each woman has a clear preference ordering of the three matchings (according to the ranking of her partners in her preference list). Now, suppose for woman $W_1$, this order is $\mathcal{M}_1 > \mathcal{M}_2 > \mathcal{M}_3$. True or false: every woman has the same preference ordering $\mathcal{M}_1 > \mathcal{M}_2 > \mathcal{M}_3$. Justify your answer.

4. **Stable Grouping.**

   (a) In a particular class, there are n students attending from each of 3 different majors (3n total students). The students are asked to split into groups of 3 such that one person from each major is in each group. Each student has a ranking of preference for the students from the other 2 majors. A student, say Anita, assigned to a group, X, will naturally prefer a different group, Y, if Anita ranks the other two members of Y higher than the other two members of X not in her major. The grouping will be unstable if also Y's member in Anita's major, say Rishi, ranks the other two members of X higher than he ranks his two current compatriots. Prove that there always exists a stable grouping of all of the students.

   Hint: try to solve this by applying what you know from the stable marriage problem. Is there something useful you can say by restricting your attention to only two of the majors. Now can you include the third major?

(b) What if there are m different majors (mn total students)? Using induction, prove that there always exists a stable grouping of all of the students.

(c) For (b), how long would the professor have to wait for the students to assort into groups? In other words, how many propose / rejects would it take to achieve a stable grouping?

5. **Modular Equations** Solve the following equations for $x$ and $y$ or show that no solution exists. Show your work (in particular, what division must you carry out to solve each case).

   (a) $5x + 23 \equiv 6 \pmod{499}$

   (b) $9x + 80 \equiv 2 \pmod{81}$

   (c) The system of simultaneous equations
   $30x + 3y \equiv 0 \pmod{37}$ and $y \equiv 4 + 13x \pmod{37}$

6. **Modular Sum of Squares** Prove that if $n = 3 \mod 4$, then $n$ cannot be the sum of the squares of 2 integers.

7. **Goldbach's Infinitude of Primes.**

   The Fermat numbers are generally described as: $F_n = 2^{2^n} + 1$ . In the 1700's, Goldbach realized that Fermat numbers held a fascinating secret.

   (a) Prove that $F_n = F_0 \times F_1 \times \cdots \times F_{n-1} + 2$.

   (b) Use (a) to show that any two Fermat numbers must be relatively prime with one another.

   (c) Use (b) to argue that there must be an infinitely many number of primes.