Problem Set 7

1. As we know, an Euler cycle in an undirected or a directed graph is a cycle that visits each edge exactly once. A *Rudrata cycle* is a cycle that visits each vertex exactly once.
   (a) Which of the five Platonic solids (search the web), considered as undirected graphs, have an Euler cycle? A Rudrata cycle?
   (b) Show that the n-dimensional hypercube has an Euler path iff n is even.
   (c) Show by induction that the n-dimensional hypercube always has a Rudrata cycle.
   (d) We know that the k-bit de Bruijn directed graph has an Euler cycle. How many Euler cycles does the 3-bit de Bruin graph (the one in the lecture notes, with 4 vertices and 8 edges) have? (The cycles $(e_1, e_2, \ldots, e_n)$ and $(e_2, e_3, \ldots, e_n, e_1)$ are considered the same.) How about the 4-bit de Bruijn graph?
   (e) Extra credit: Can you generalize to n-bit de Bruijn graphs and prove?

2. Recall that an n-dimensional hypercube contains 2^n vertices, each labeled with a distinct n bit string, and two vertices are adjacent iff their bit strings differ in exactly one position.
   (a) The hypercube is a popular architecture for parallel computation.Let each vertex of the hypercube represent a processor and each edge represent a communication link. Suppose we want to send a packet for vertex x to vertex y. Consider the following "bit-fixing" algorithm:

   > In each step, the current processor compares its address to the destination address of the packet. Let's say that the two addresses match up to the first k positions. The processor then forwards the packet and the destination address on to its neighboring processor whose address matches the destination address in at least the first $k+1$ positions. This process continues until the packet arrives at its destination.

   Consider the following example where $n = 4$: Suppose that the source vertex is (1001) and the destination vertex is (0100). Give the sequence of processors that the packet is forwarded to using the bit-fixing algorithm.
   (b) In general, for an arbitrary source vertex and arbitrary destination vertex, how many edges must the packet traverse under this algorithm? Give an exact answer in terms of the n-bit strings labeling source and destination vertices.
   (c) Consider any two vertices x and y in the hypercube. Consider the graph $G = (V, E)$ where E is the set of all edges on shortest paths between x and y and V is the set of all vertices on shortest paths between x and y.
   Consider the following example where $n = 3$: Suppose that x is (010) and y is (100). What is the length of the shortest path between x and y? Explicitly show the sets V and E.
   (d) In general, for arbitrary x and y, what are the sizes of V and E? Express your answer in terms of the n-bit strings labeling of x and y.

3. Consider a deck of cards with 52 cards in 13 values and four suits. How many sets of three cards drawn from this deck have:
   (a) all three the same value.
   (b) all three the same suit.
   (c) two of them have the same value, different from the third.
   (d) they have "cyclically consecutive values," such as 9, 10, J, or K, A, 2.
   (e) How many sets of three cards are there anyway?