CS 70 FALL 2006 — DISCUSSION #14

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1. Administrivia

Course Information

- Homework # 13 is due today.
- Next week we will post exercises and sample final exams by Friday. There will be a review session on Sunday, time TBA in Soda Hall.
- Notes on diagonalization and the Incompleteness Theorem will be posted today.

2. Countablity

Let us recall a few definitions.

Definition 1 : A set S is countable if and only if there is an onto (*surjective*) function from $\mathbb{N}$ to S.

Definition 2 : A set S is countably infinite if and only if S is countable and infinite.

The following exercises explore the meaning of these definitions:

Exercise 1. Consider the following alternative definition of countability: S is countable if and only if there is an injective mapping from S to $\mathbb{N}$. Is this equivalent to Definition 1 above? Prove or disprove.

Exercise 2. Show that if S is countably infinite there is a bijection between $\mathbb{N}$ and S.

Exercise 3. Challenge Let A and B be two finite sets. Suppose there is a surjection $f : A \to B$ and a surjection $g : B \to A$. Prove that it is possible to construct a bijection between A and B. Does your argument go through if A and B are infinite?

Suppose you are asked to prove that a set S is countable. How do you go about it? Here are two strategies:

- Use the definition, try to construct a surjection from $\mathbb{N}$ to S.
- Consider a set T you know to be countable ($\mathbb{Q}, \mathbb{N}^2, \cdots$). Try to construct a surjection from T to S (why does this satisfy the definition?).

Exercise 4. Make use of the second strategy cleverly to prove the following:

- $\mathbb{N}^n$ is countable for all n.
- $\bigcup_{i \in N} \mathbb{N}^i$ is countable.

Date: December 8, 2006.

3. Uncountability

Definition 3: A set S is uncountable if it is not countable.

In class, we saw how to use Cantor's diagonalization method to prove that the set $\mathbb{R}$ of real numbers is uncountable.

Exercise 5. "Any subset S of $\mathbb{R}$ such that $S - \mathbb{Q} \neq \emptyset$ is uncountable". Prove or disprove.

As in the countable case, there are two main strategies we can follow when trying to prove that a set is uncountable:

- Directly prove it is not countable using diagonalization.
- Construct a surjection from S to $\mathbb{R}$. Use the definition to prove this suffices.

Exercise 6. Use the second strategy to prove that $2^{\mathbb{N}}$ is uncountable. You saw this in class.

Exercise 7. Prove that $\mathbb{C}$ is uncountable.

Exercise 8. Prove that the set of functions f s.t. $f : \mathbb{N} \to \{0, 1\}$ is uncountable.