CS 70 FALL 2006 — DISCUSSION #2

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1. Administrivia

(1) Course Information
- The second homework is due September 8^{st} at 4pm in 283 Soda Hall

(2) Discussion Information
- If you have a clash, it is OK to attend a section different to your enrolled/wait-listed one. Just be sure to show up so that we can 'assign' you somewhere based on the roles taken in sections in the first few weeks.

2. Bad Proofs

Consider the following false statement and its (necessarily!) erroneous proof.

Theorem 1. *2=1.*

Proof. Take any $a, b \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that $a = b$

$$\Rightarrow \quad a = b \tag{2.1}$$

$$\Rightarrow \quad a^2 = ab \tag{2.2}$$

$$\Rightarrow \quad a^2 - b^2 = ab - b^2 \tag{2.3}$$

$$\Rightarrow \quad (a+b)(a-b) = (a-b)b \tag{2.4}$$

$$\Rightarrow \quad a + b = b \tag{2.5}$$

$$\Rightarrow \quad 2b = b \tag{2.6}$$

$$\Rightarrow \quad 2 = 1\ . \tag{2.7}$$

$\square$

Exercise 1. What went wrong with this proof? What lessons can be learned? $\square$

Exercise 2. Given $a, b \in \mathbb{R} - \{0\}$ and $ab > 1$, a student concludes $a > 1/b$. Is this always true? If not, where did the student go wrong? $\square$

3. Biconditional Proofs

Friday's lecture introduced a number of types of proofs, including direct proofs and proof by contraposition which both aim to prove a statement of the form $P \Rightarrow Q$. Often our goal will *additionally* be to prove the converse $Q \Rightarrow P$ – that is we are to prove $P \Leftrightarrow Q$.

Theorem 2. *n is odd iff n^2 is odd, for each $n \in \mathbb{N}$.*

Date: September 9, 2006.

The authors gratefully acknowledge Chris Crutchfield and Amir Kamil for the use of their previous notes, which form part of the basis for this handout.

Exercise 3. Consider Theorem 2.

(i) Begin by proving the forward direction (n odd implies n^2 odd). easy proof by algebra
(ii) Carefully prove the theorem with a simple modification to part (i).
(iii) Appeal to the equivalence of an implication and its contrapositive to prove the corollary[1] that "n is even iff n^2 is even, for each $n \in \mathbb{N}$.

□

4. Different People, Different Proofs

Consider the following theorem.

Theorem 3. *Given a sequence of real numbers $x_0 = 1$ and $x_1, x_2, x_3, x_4, x_5 \geq 1$, the following holds true: if $x_5 > 35$, then $\exists i \in \{0, 1, 2, 3, 4\}$ such that $\frac{x_{i+1}}{x_i} > 2$.*

You can prove this in any of the three ways, you learnt in class: direct proof, proof by contrapositive and proof by contradiction.

Exercise 4. Prove the theorem in each of the three ways. Which one was easier? Which one more natural to you?

5. Algebraic Inductions

Let's try some practice induction problems that look like those covered in lecture this week.

Exercise 5. Prove that $1^2 + 3^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$. □

Exercise 6. (i) A geometric series is an infinite sum of the form $1+x+x^2+x^3+x^4+\ldots$ for some real x. Prove that the series' partial sum $1+x+x^2+\ldots+x^n$ equals $\frac{x^{n+1}-1}{x-1}$. Many times a guess is good and then you can use induction to actually prove it.
(ii) An arithmetic series is a series of the form $\sum_{i=1}^{\infty} a_k$ where $a_{k+1} = a_k + d$ for each positive integer k and $a_1, d \in \mathbb{R}$ are picked arbitrarily. Find the closed-form partial sum of this series and prove your result by induction.

□

6. Non-Algebraic Inductions

Many algebraic induction problems have been explored in class and in the homework set already. The following two exercises demonstrate the broader applicability of induction as a proof technique in discrete math.

Last week we proved De Morgan's Laws intuitively for arbitrary universes and intuitively & via truth table for two-element universes.

Exercise 7. Recall De Morgan's binary-universe laws $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ and $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$, and the corresponding quantified generalizations $\neg \forall x, P(x) \equiv \exists x, \neg P(x)$ and $\neg \exists x, P(x) \equiv \forall x, \neg P(x)$. Prove the quantified versions formally for finite universes, using induction. □

Induction is a powerful proof technique in many geometric problems.

[1] A 'corollary' is a result that immediately follows from a proven result.

Exercise 8. An *arrangement* of lines in the plane $\mathbb{R}^2$ is a set of lines satisfying the property that no point in the plane $x \in \mathbb{R}^2$ is the intersection of 3 or more lines

(i) Draw some example sets of lines that are/aren't legal arrangements.

An arrangement divides the plane up into *cells* or polyhedral regions that have segments of the arrangement's lines as borders. Two cells are called *neighbors* if they share a border segment (shared border points don't count). A 2-coloring of the cells of an arrangement is an assignment of 'black' or 'white' to each cell such that no neighboring cells share the same color.

(ii) Use induction to prove that an arrangement's cells can always be 2-colored.

□

7. Strong Induction: Sums of Fibonacci & Prime Numbers

Many of you may have heard of the Fibonacci sequence. We define $F_1 = 1, F_2 = 1$, and then define the rest of the sequence recursively: for $k \geq 3$, $F_k = F_{k-1} + F_{k-2}$. So the sequence ends up looking like:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

While not all positive integers are Fibonacci (e.g. 4), surprisingly we can express any positive integer as the sum of distinct terms in the Fibonacci sequence.

Theorem 4. *Every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.*

Proof. Let $P(n)$ be the statement that n can be expressed as the sum of distinct terms in the Fibonacci sequence. We begin with the base case $n = 1$. Since 1 is a term in the Fibonacci sequence (namely F_1), then $P(1)$ is true.

Now we proceed to the inductive step. We wish to show that $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$. So assume that $P(1), P(2), \ldots, P(n)$ hold. Now we consider $n+1$. There are two cases:

(1) $n + 1$ is itself a Fibonacci number.
(2) $n + 1$ is not a Fibonacci number.

If the former holds, then we're done. If the latter holds, then there must exist some positive integer k such that

$$F_k < n + 1 < F_{k+1}.$$

Since $F_k < n+1$, we may decompose $n+1$ into $F_k + (n+1-F_k)$. But by definition, $(n + 1 - F_k) < n + 1$ so by the inductive hypothesis we know that $P(n + 1 - F_k)$ is true, hence it may be expressed as such:

$$n + 1 - F_k = F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

where the subscripts are distinct. Moreover, since $n+1-F_k < F_k$ (since $n+1 < F_{k+1}$ implies $n + 1 - F_k < F_{k-1} < F_k$) it is not possible that any of the F_{i_j} could be equal to F_k. Therefore we have

$$n + 1 = F_k + F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

and $P(n + 1)$ holds. Thus by strong induction, $P(n)$ holds for all $n \geq 1$. □

Similarly one might attempt to prove the analogous result with primes (repeats allowed).

Exercise 9. Prove that all integers greater than one can be expressed as the sum of primes. $\square$