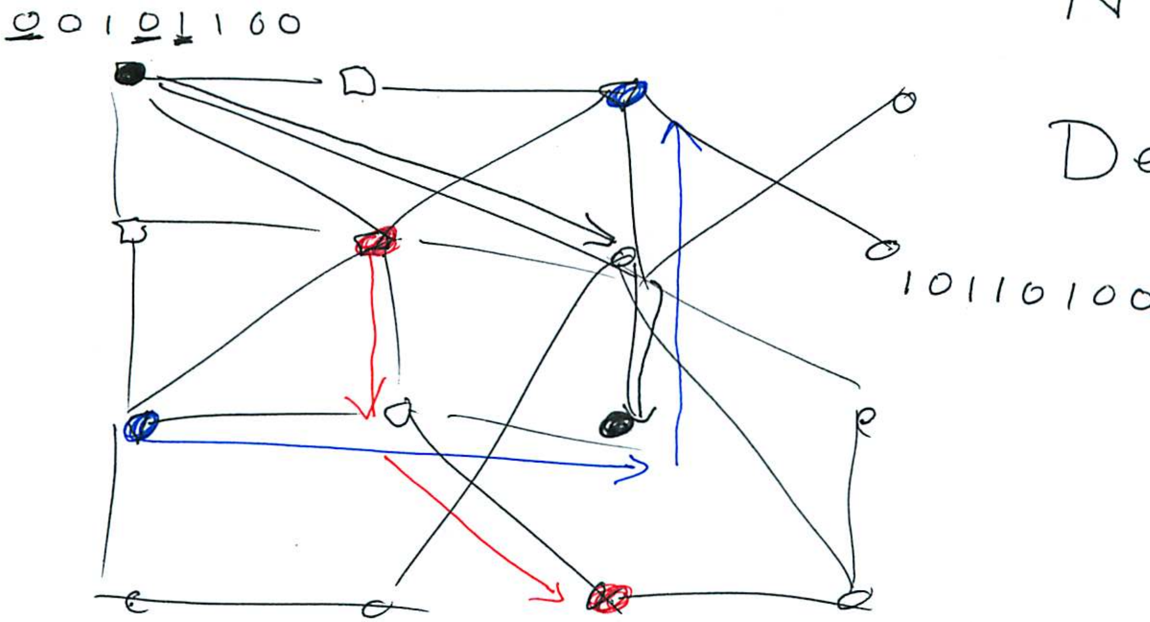
Hypercubes

N processors.

Design a graph on N vertices



How costly? # edges.

Minimal Requirement : ~~H~~ G is connected.

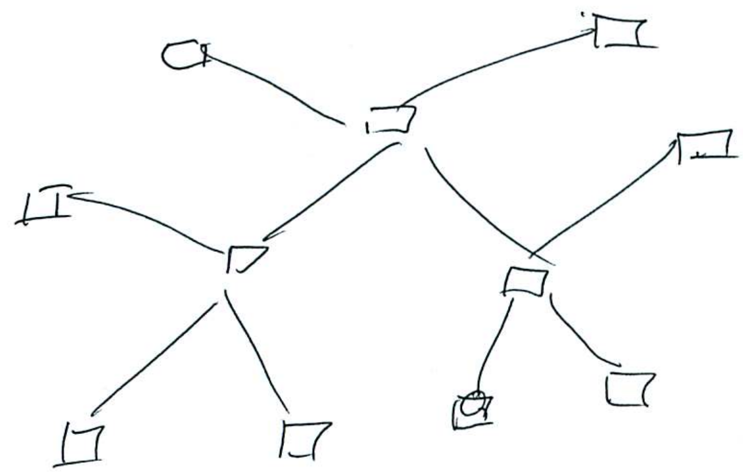
How many edges?

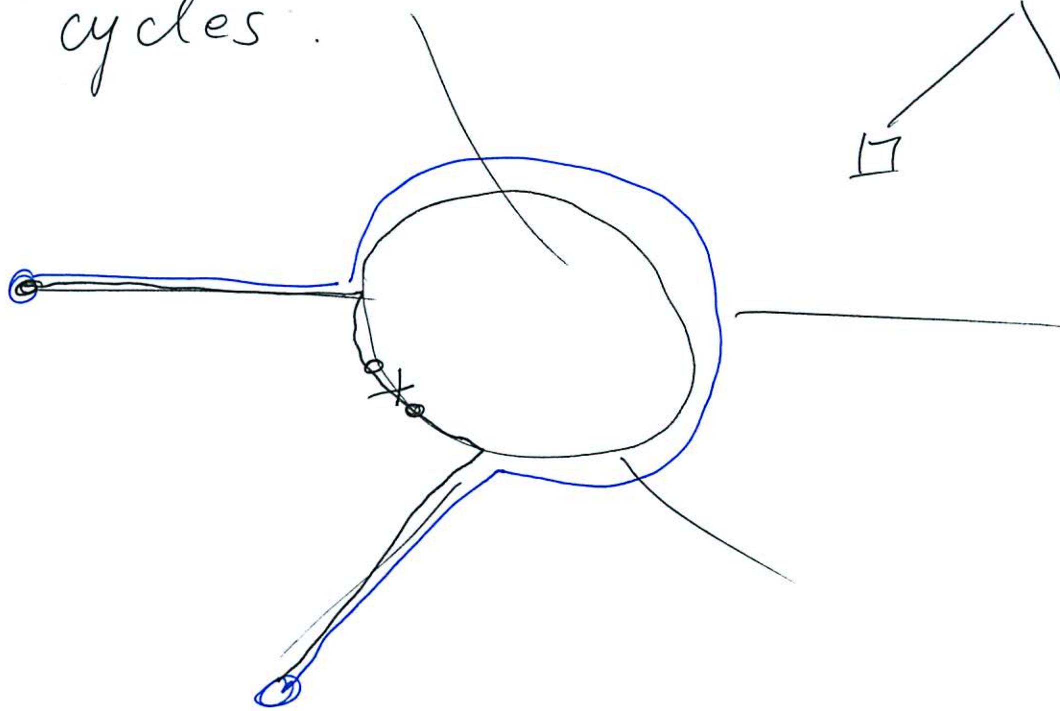
N vertices.

<u>Ans:</u> N-1 edges.

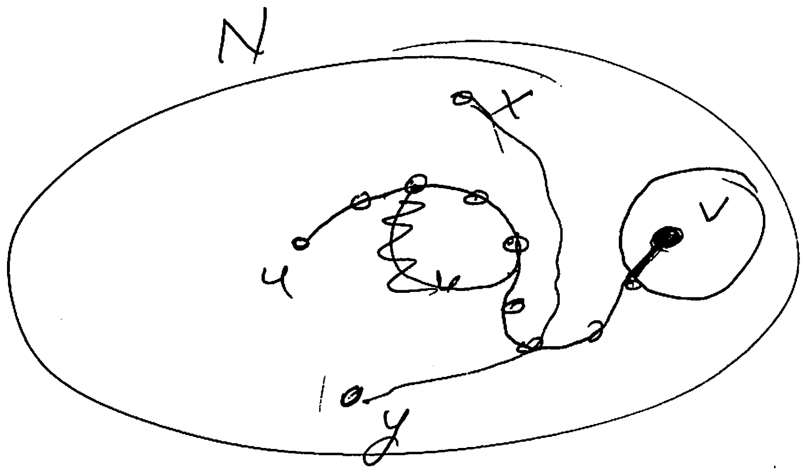
<u>Tree</u>

No cycles.





Connected, acyclic graph on N vertices has N-1 edges.



Start at u & walk until ? can't loop back get stuck.

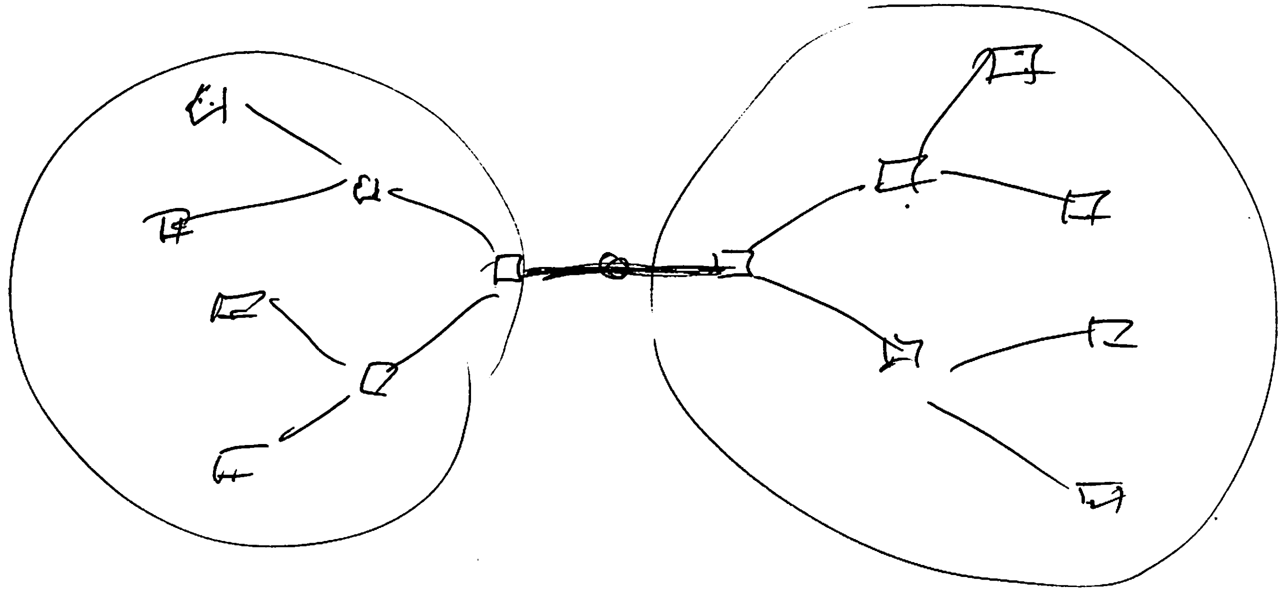
Find v : deg(v) = 1.

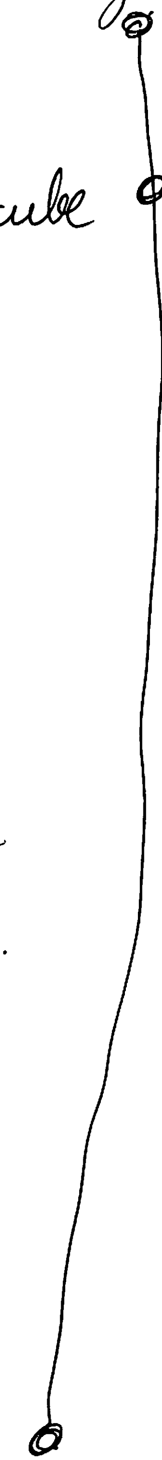
Remove v & edge

Resulting graph has N-1 vertices connected acyclic.

Apply induction hypothesis.

Tree   N−1 edges.



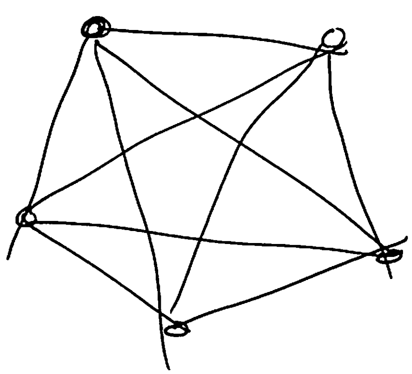


Hypercube

No congestion. Reliability.

No bottlenecks.

Potential solution: Complete graph.



$$\deg(v) = N-1$$

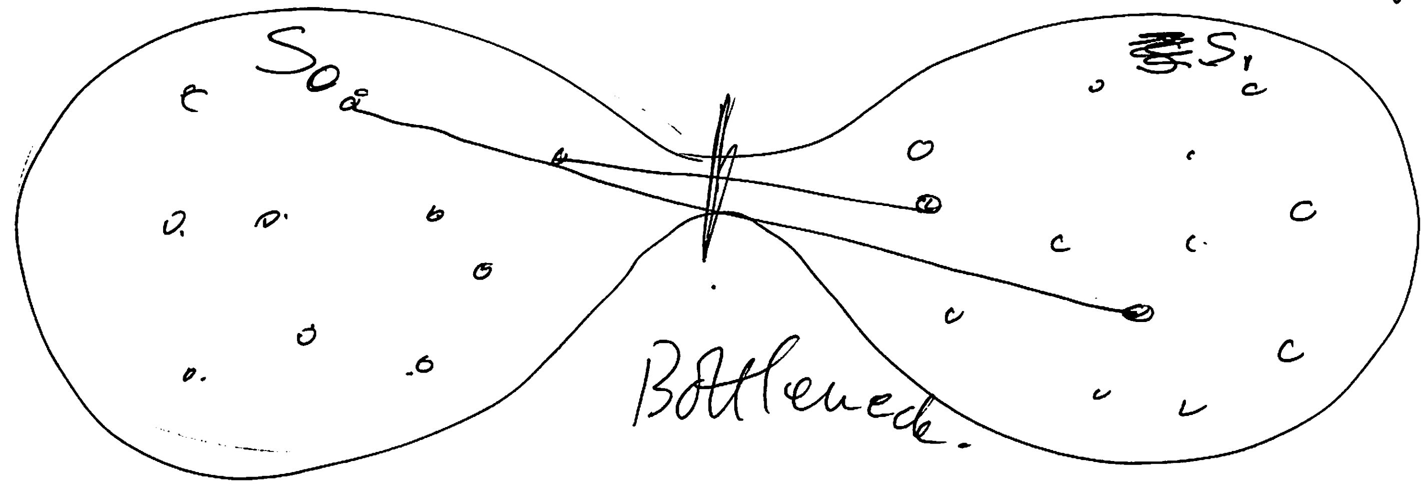
$$\sum_{v} \deg(v) = 2|E|.$$

$$N(N-1) = 2|E|$$

$$|E| = \frac{N(N-1)}{2}.$$

V

$V = S_0 \cup S_1$.



$$\min \frac{\#\text{ edges cut}}{\#\text{ vertices on smaller side}}$$

Hypercube: $\#\text{ edges cut} \geq |S_0|$
$= \#$ vertices on on smaller side

Proof by induction.

Hypercube

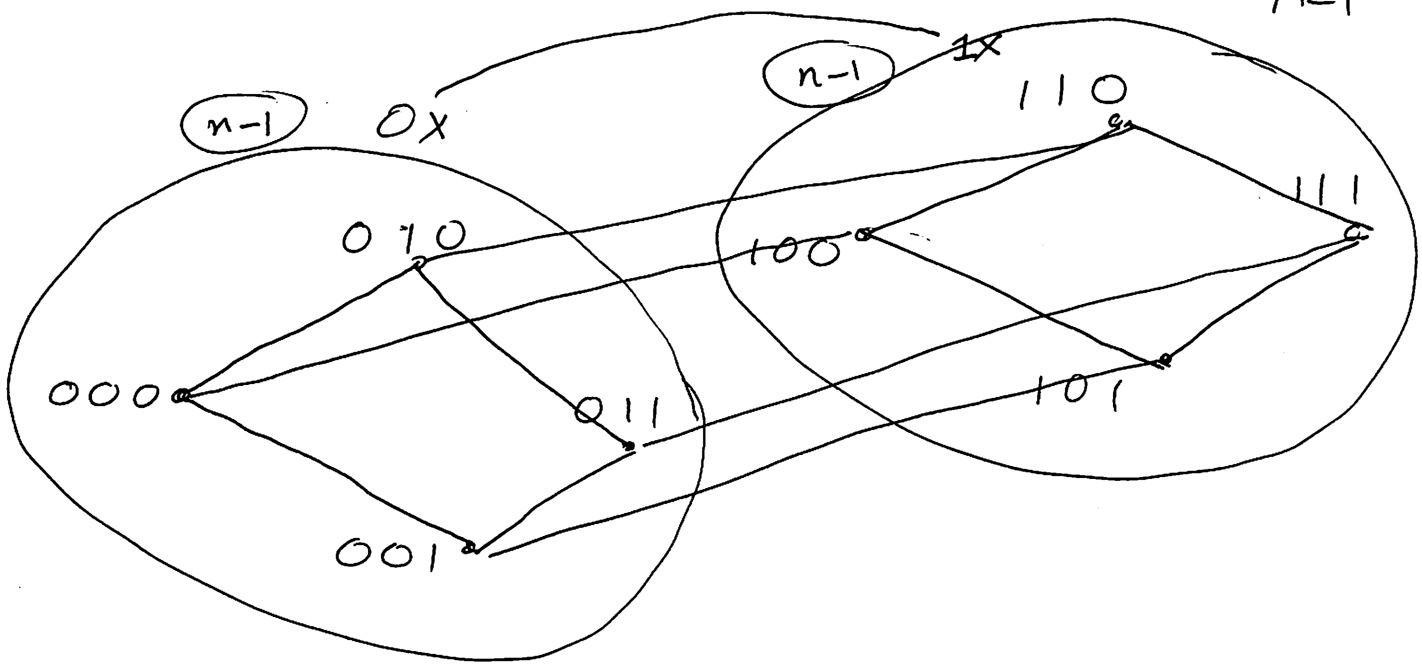
$N = 2^n$

Example: $n = 3$.

$\deg(v) = n = \log_2 N$

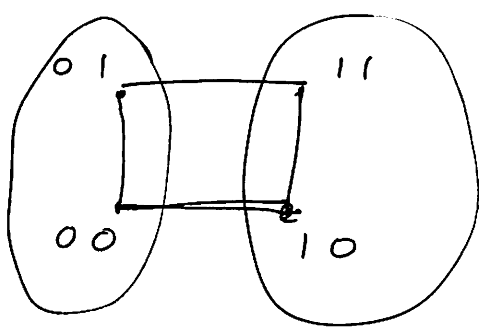
$\sum_v \deg(v) = nN = 2|E|$.

$|E| = \frac{N \log N}{2}$



$n = 2$

$N = 2^2 = 4$



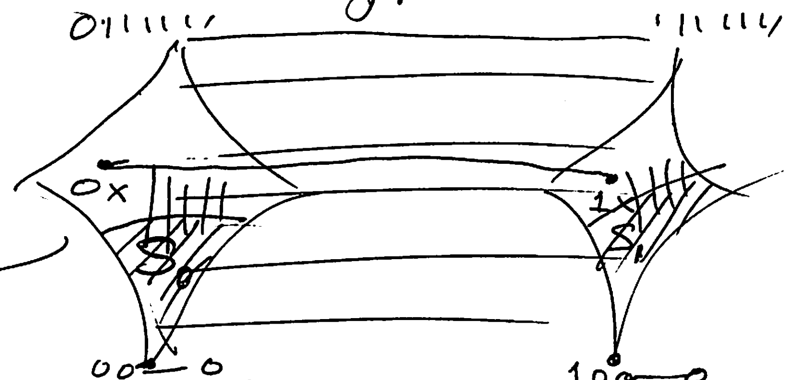
$n = 1$

$N = 2^1$

$n = 0$

$N = 2^0 = 1$

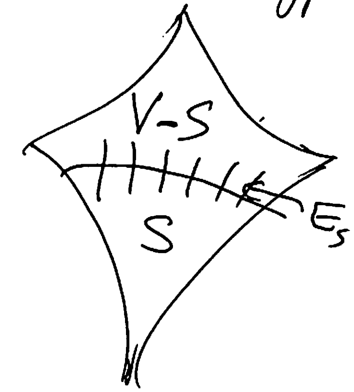
n-dim hypercube $\qquad N = 2^n$



$n-1$ dim $\qquad$ $n-1$ dim.

0-subcube. $\qquad$ 1-subcube.

Induction Hypothesis:



$n-1$ dim.

$$|S| \le \frac{2^{n-1}}{2} = 2^{n-2}$$

$$|E_s| \ge |S|.$$

$$S = S_0 \cup S_1$$

$$|S| \le 2^n/2$$

<u>Case 1</u>: $|S_0| \le 2^{n-2}$ & $|S_1| \le 2^{n-2}$.

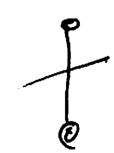
~~Ed~~ crossing edges in 0-subcube $\ge |S_0|$

" $\quad$ " $\quad$ " $\quad$ 1-subcube $\ge |S_1|$.

$\therefore$ # crossing edges $\ge |S_0| + |S_1| = |S|$.

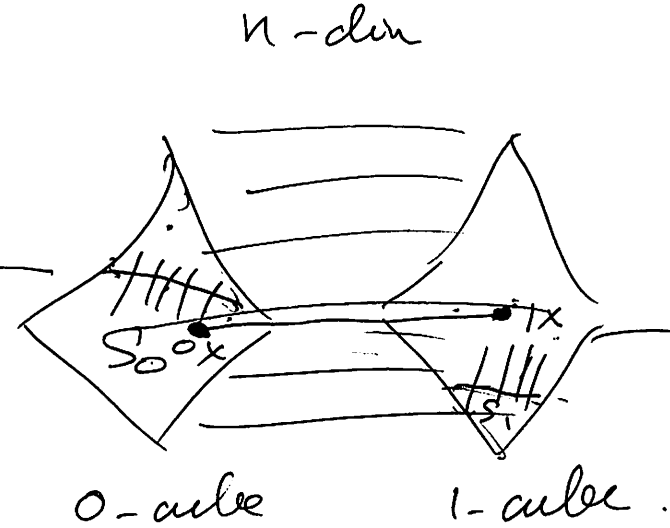
$$\overset{\shortparallel}{|E_s|}$$

$n=1$



Case 2 :

$S = S_0 \cup S_1$



Case 2 : $|S_0| \geq 2^{n-2}$

Means $|S_1| \leq 2^{n-2}$.

crossing edges in 1-subcube $\geq |S_1|$.

crossing edges in 0-subcube $\geq 2^{n-1} - |S_0|$.

sum $\geq 2^{n-1} + |S_1| - |S_0|$.

crossing edges going between 0-subcube & 1-subcube $\geq |S_0| - |S_1|$.

crossing edges $\geq \underline{2^{n-1}} + \cancel{|S_1|} - \cancel{|S_0|} + \cancel{|S_0|} - \cancel{|S_1|} \geq |S|$.