Conditional Probability

Coin Flipping: 10 times Fair coin.
First 9 flips are H's.
P[10th flip H] = \frac{1}{2}.

Deck of cards B ↔ H R ↔ T.

\[ \theta \]

\[ P[1st \text{ card is } B] = \frac{1}{2} \]
\[ P[2nd \text{ card is } B] = \frac{1}{2} \] — Assume we didn't look at 1st card.

\[ P[2nd \text{ card is } B \text{ if } 1st \text{ card is } B] = ? = \frac{25}{51} \]

\[ P[27th \text{ card is } B \text{ if first 26 are all } B] = 0 \]
Hot Hand

\[ P \left[ \text{Basket} \mid \text{previous attempt} \right] \]

\[ P \left[ \text{Basket} \right] \]
\[ \Omega = \text{sample space} \]

\[
P(A) = \frac{|A|}{|\Omega|}
\]

\[
P(B) = \frac{|B|}{|\Omega|}.
\]

\[P(\text{A occurs given B occurs}) = \frac{|A \cap B|}{|B|}\]

\[P[A \mid B] = \frac{P[A \cap B]}{P[B]}.
\]
4 cards  2R  2R.

\[ P[1^{st \text{ card}} = B] = \frac{3}{6} = \frac{1}{2}. \]

\[ P[2^{nd \text{ card}} = B] = \frac{3}{6} = \frac{1}{2} \]

\[ P[2^{nd \text{ card}} = B/1^{st \text{ card}} = B] = \frac{1}{3} \]

\[ P[3^{rd \text{ card}} = B/1^{st} = R \text{ and } 2^{nd} = B] = 0 \]

\[ P[2^{nd \text{ card}} = B/1^{st} = R \text{ and } 3^{rd} = B] = \frac{1}{2} \]
Independence:

A & B are independent events

\[
\begin{align*}
P[A/B] &= P[A] \\
\frac{P[A \cap B]}{P[B]} &= P[A] \\
P[A \cap B] &= P[A] \cdot P[B].
\end{align*}
\]

Biased coin  \( P[H] = p \)  \( P[T] = 1 - p \).

Flip coin n times  \( P \left[ \underbrace{H \cdots H}_k, \underbrace{T \cdots T}_{n-k} \right] \)

\[
\begin{align*}
P[H] &= p^k (1-p)^{n-k}.
\end{align*}
\]

\( P[HH] \)  \( A = 1^{st} \) toss \( H \).  \( P[A] = p \).

\( B = 2^{nd} \) toss \( H \).  \( P(B) = p \).

\( P[A \cap B] = P[A] \cdot P(B) = p^2. \)
\[
\frac{|A|}{|\Omega|} = p
\]
\[
\frac{|B|}{|\Omega|} = q.
\]
Bayes Rule

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}
\]

\[A = \text{Kidney cancer}\]

\[B = \text{Microscopic Hematuria (MH)}\]

\[P(B|A) = 1\]

\[P(A) = 0.001\]

\[P(B) = 0.05\]

\[P[A|B] = \frac{P(B|A)P(A)}{P(B)} = \frac{1 \times 0.001}{0.05} = 0.02\]
Choose a coin at random from A and it turns out to be gold. What is the chance that A contained 2 gold coins?

Let $X = A$ contains 2 gold coins.

Let $Y = \text{choose a gold coin from A}$.

$$P[X|Y] = \frac{P[Y|X]P[X]}{P[Y]}$$

$P[Y|X] = 1$ \quad $P[X] = \frac{1}{2}$ \quad $P[Y|\overline{X}] = \frac{1}{2}$ \quad $P[\overline{X}] = \frac{1}{2}$

Rule of total Prob: $P[Y\cap X] + P[Y\cap \overline{X}]$

$$P[Y] = P[Y|X]P[X] + P[Y|\overline{X}]P[\overline{X}]$$
\[ P(x|y) = \frac{P(y|x)P(x)}{P(y|x)P(x) + P(y|x')P(x')} = \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}. \]
Monte Hall Paradox

Contestant picks a door.
Host opens one of the remaining doors to reveal goat.
Contestant is allowed to switch.

\[ P(\text{contestant wins if switch}) = \frac{1}{3} \Rightarrow P(\text{win if switch}) = \frac{2}{3}. \]