Lecture 16
CS70 Fall 2013

Conditional Probability II : Intersection & Union of Events.

A, B independendent iff $P(A \cap B) = P(A) \cdot P(B)$.

$$P(A) \underset{\text{indep}}{=} P(A|B) \underset{\text{Def}}{=} \frac{P(A \cap B)}{P(B)}$$

A, B, C multually indep.

$$P(A) = P(A \mid B \cap C) = \frac{P(A \cap (B \cap C))}{P(B \cap C)}$$

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C).$$

<u>Coin flips</u> $P(HTH)$ $\qquad P(H) = p.$

$= p(1-p)p$

Pairwise Independence

fair coin

1 wp $\frac{1}{2}$ 0 wp $\frac{1}{2}$. X

$x + z \pmod 2 = y$

Z $x + y \pmod 2$.

A = first flip is Hs.

B = second flip is Hs.

C = both flips same.

HH or TT.

Are B, C independent.

$$P(B) = \frac{1}{2}$$
$$P[C] = \frac{1}{2}$$

$$P(B \cap C) = \frac{1}{4}$$

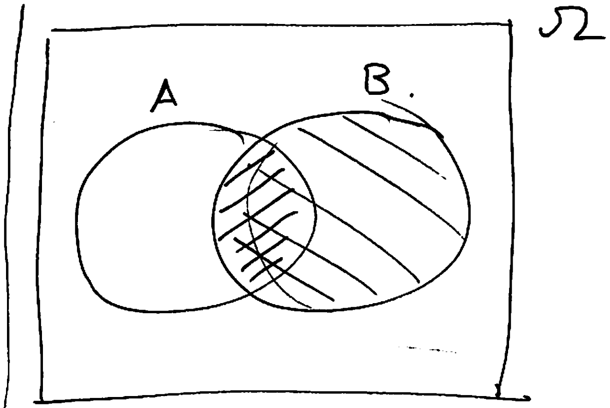
$$B \cap C = \{ HH \}$$

A, C also independent.

A, B, C mutually indep? Not!

Intersection of Events

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \implies \underline{P(A\cap B) = P(B)\,P(A|B)}.$$



$$\underbrace{\frac{|\cancel{B}|}{|\Omega|}}_{P(B)} \cdot \underbrace{\frac{|A\cap B|}{|\cancel{B}|}}_{P[A|B]}.$$

$$P[A\cap B\cap C] = P[A|B\cap C]\,P[B|C]\,P(C).$$

$A_1, A_2, \dots, A_n.$

$$\bigcap_{i=1}^{n} A_i = A_1\cap A_2 \cap \cdots \cap A_n.$$

$$P\left[\bigcap_{i=1}^{n} A_i\right] = P\left[A_1 \,\middle|\, \bigcap_{i=2}^{n} A_i\right] P\left[A_2 \,\middle|\, \bigcap_{i=3}^{n} A_i\right] \cdots P[A_n]$$

Law of successive conditioning.

1st rule of counting.

__example:__ coin flips.

$$P[\text{flush}] = 4 \times \underbrace{\frac{13}{52}}_{\substack{1^{st}\text{ card}\\ \text{spade}}} \times \underbrace{\frac{12}{51}}_{\substack{2^{nd}\text{ card}\\ \text{spade} \mid \\ 1^{st}\text{ is spade}}} \times \underbrace{\frac{11}{50}}_{\substack{3^{rd}\text{ spade} \mid \\ 1^{st}\,\&\,2^{nd}\\ \text{spades.}}} \times \frac{10}{49} \times \cdots \times \text{—}$$

Proof: Induction on n.

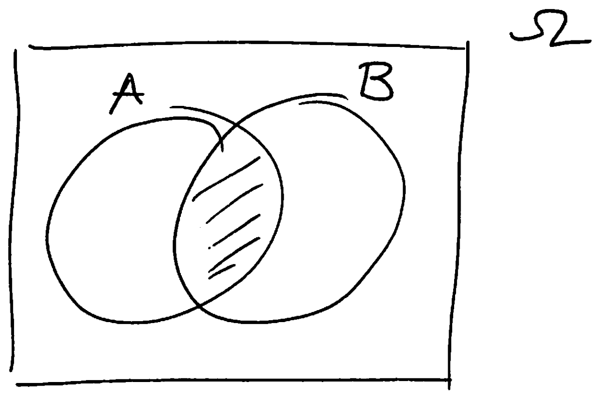
Holds for $n-1$.

$$P\Big[\bigcap_{i=1}^{n} A_i\Big] = P\Big[\overset{A}{\overset{\shortparallel}{A_1}} \cap \overset{B}{\overset{\shortparallel}{\Big(\bigcap_{i=2}^{n} A_i\Big)}}\Big] = \overbrace{P\Big[A_1 \Big| \bigcap_{i=2}^{n} A_i\Big]}^{A|B} \cdot \underbrace{\overset{B}{P\Big[\bigcap_{i=2}^{n} A_i\Big]}}_{\text{Apply induction hypothesis.}}$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = A_1 \cap [A_2 \cap \cdots \cap A_n]$$

Union of Events

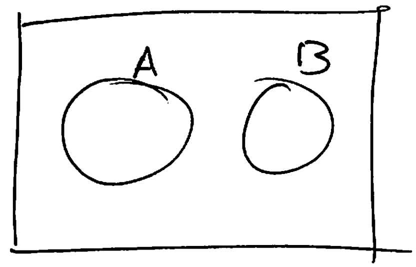
$P[A \cup B]$



Disjoint : $A \cap B = \{\}$

$P[A \cup B] = P[A] + P[B]$.

$P[H] + P[T] = 1$.

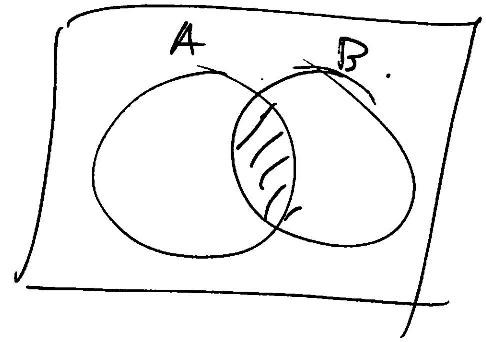


$P[\text{flush}] = 4 \times P[\text{spade flush}]$.

$P(A \cup B) \leq P(A) + P(B)$ Union Bound.

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$.

$$P[A \cup B) = P[A] + P[B] - P(A \cap B)$$



A = H's toss a fair coin.

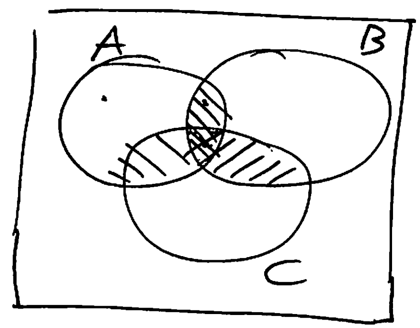
B = pair roll two dice.

$$P[A] = \tfrac{1}{2} \qquad P[B) = 1 \times \tfrac{1}{6} = \tfrac{1}{6}.$$

$$P[A \cap B] = \tfrac{1}{2} \times \tfrac{1}{6}. = \tfrac{1}{12}$$

$$P[A \cup B) = \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{12} = \frac{6+2-1}{12} = \frac{7}{12}.$$

$$P[A \cup B \cup C] = P[A] + P[B) + P[C] - P[A \cap B) - P[B \cap C) - P[A \cap C] + P[A \cap B \cap C].$$



$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = \sum_{i=1}^{n} P[A_i] - \sum_{\{i,j\}} P[A_i \cap A_j] + \sum_{\{i,j,k\}} P[A_i \cap A_j \cap A_k] - \sum_{\{i,j,k,l\}} P[A_i \cap A_j \cap A_k \cap A_l] + \cdots + (-1)^{n+1} P[A_1 \cap A_2 \cap \cdots \cap A_n]$$

Inclusion-Exclusion formula.

n letters $\quad n$ envelopes.

P[no letter in matchg envelope].

A_i = i^{th} letter ~~ends up~~ in i^{th} envelope.

$P[\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n]$ ✓.

$\Omega \quad +$

$P[A_1 \cup A_2 \cup \cdots \cup A_n] \Big\} = 1.$

P[exactly K letters in correspondg envelope] $= \dfrac{1}{K!}$

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \cdots \pm \frac{1}{n!}$$

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \pm \frac{1}{n!} \approx e^{-1} = \frac{1}{e}.$$

$e = 2.718\ldots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad x = -1.$$