Random Variables - Expectation.

Flip a coin n times (10,000). Number of H's.

\[ X = \# \text{ of H's.} \]

\[ X \in \{0, 1, 2, \ldots, 5000, 9999, 10000\}. \]

\[ X \text{ is a random variable + probability of taking on each value.} \]

\[ P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \]

n letters and n envelopes. \( n=20 \)

\[ Y = \# \text{ letters that end up in their envelopes.} \]

\[ Y \in \{0, 1, 2, \ldots, 20\}. \]

\[ P[Y = k] = p^k. \]

Expected value of r.v. \( E[X] = 5000 \)

Average of a large number of trials.
\[ P[H] = P \]

\[ P[X=k] = \binom{n}{k} P^k (1-P)^{n-k} \]
Sample Space \( \mathcal{S} \): \[ \begin{array}{c}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 1 & 2 \end{array} \]

\[6 = 3! \]

\[Y = 3 \quad Y = 1 \quad Y = 0\]

\( Y \) is a function that maps \( \mathcal{S} \) to \( \mathbb{Z} \).

**Definition**

Integer random variable \( X \) is a function that maps \( \mathcal{S} \) to \( \mathbb{Z} \).

\[X : \mathcal{S} \to \mathbb{Z} \]

**Events**: \( X = k \) is an event.

\[\{X = k\} = \{ \omega \in \mathcal{S} : X(\omega) = k \} \]

such that
Flip biased coin \( P(H) = p \) until \( n \)th \( H \)s.

How long before \( n \)th \( H \)s?

\[ W_n \]

\[ P[W_n = n] = (1-p)^{n-1} \cdot p \]

\[ \sum_{n=1}^{\infty} (1-p)^{n-1} \cdot p = 1 \]

\[ p \sum_{k=0}^{\infty} (1-p)^{k} = \frac{1}{1 - (1-p)} = p \cdot \frac{1}{p} = 1 \]
\[ W_k = \text{waiting time for } k^{th} \text{ H's.} \]

\[ P[W_k = n] = \binom{n-1}{k-1} p^k (1-p)^{n-k} \]

\[ \sum_{k=1}^{n} P[W_k = n] = 1. \]

\[ \sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k (1-p)^{n-k} = 1. \]
Flip biased coin \( n \) times

\[ S_n = \# \text{ H's}. \]

\[ P[S_n = k] = \binom{n}{k} p^k (1-p)^{n-k}. \]

\[ \sum_{k=0}^{n} P[S_n = k] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1. \]

**Binomial Theorem**

\[ 1^n = [p + (1-p)]^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \]

\[ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \]
\( X \)

\[
P[X = 6] = P_0 = \frac{2}{6}
\]

\[
P[X = 1] = P_1 = \frac{8}{18} = \frac{4}{9}
\]

\[
P[X = 2] = P_2 = \frac{0}{6}
\]

\[
P[X = 3] = P_3 = \frac{1}{6}
\]

Distribution of \( X \)

\[
\text{Expected value of } X = E[X] = \sum_{n=-\infty}^{\infty} n \cdot P[X = n]
\]
Envelope example with \( n = 3 \).

\[
\mathbb{E}[X] = 0 \cdot \frac{2}{6} + 1 \times \frac{3}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6}.
\]

\[
= \frac{3}{6} + \frac{3}{6} = 1.
\]

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General \( n \).

\[
\mathbb{P}[X = k] \text{ hard!}
\]

\[
\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}[X = k]
\]

Linearly of expectation:

Define random variable \( X_i \) if letter is in envelope \( i \).

\( X_i \) is an indicator r.v.

\( X = X_1 + X_2 + \ldots + X_n. \)

\[
\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n].
\]

\[
\frac{16}{64} = \frac{1}{4}.
\]
\[ E(X_i) = 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n} = \frac{1}{n} \]  

\[ E \left( X_i \right) = \frac{1}{n} \]  

\[ E(X) = n \times \frac{1}{n} = 1 \]
Baseball cards:

100 baseball cards.

\[ X = \text{# distinct cards in 120 days.} \]

\[ E[\text{# distinct cards you collect in } 120 \text{ days}] \]

\[ X_i = \begin{cases} 1 & \text{if collect } i^{th} \text{ player} \\ 0 & \text{otherwise}. \end{cases} \]

\[ P[X_{18} = 1] = E[X_{18}] \]

\[ X = X_1 + X_2 + \ldots + X_{100} \]

\[ E[X] = 100 E[X_{i}] = 100 P[X_{i} = 1] \]

\[ P[X_i = 0] = \left( \frac{99}{100} \right)^{120} \]

\[ P[X_i = 1] = 1 - \left( \frac{99}{100} \right)^{120} \]
Random permutation \( 3, 1, 2, 5, 4 \)

\( X = \text{Cant} \) # inversions.

\[ \mathbb{E}[X] \]

\[ X_{i, j} = \begin{cases} 1 & \text{if } i > j \text{ inverted} \\ 0 & \text{otherwise} \end{cases} \]

\[ X = \sum_{i \neq j} X_{i, j} \]

\[ \mathbb{E}[X] = \binom{n}{2} \mathbb{E}[X_{i, j}] = \frac{n(n-1)}{2} \times \frac{1}{2} = \frac{n(n-1)}{4} \]

\[ \mathbb{P}(X_{i, j} = 1) = \frac{1}{2} \]

\[ \mathbb{E}[X_{i, j}^2] = \frac{3}{2} \]
$X, Y$ are r.v. on some sample space $\Omega$ then $E[X+Y] = E[X] + E[Y]$. 