Random Variables - Expectation.

$P[H] = p$.

Flip a coin n times (10,000). Number of H's.

$X = \#$ of H's.

$X \in \{0, 1, 2, \ldots, 5000, \quad, 9999, 10000\}$.

X is a integer random variable + probability ~~of~~ taking on each value.

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

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n letters and n envelopes. $\underline{n=20}$

$Y = \#$ letters that end up in their envelopes.

$Y \in \{0, 1, 2, \ldots, 20\}$.

$P[Y = k] = p_k$.

Expected value of r.v. $E[X] = 5000$

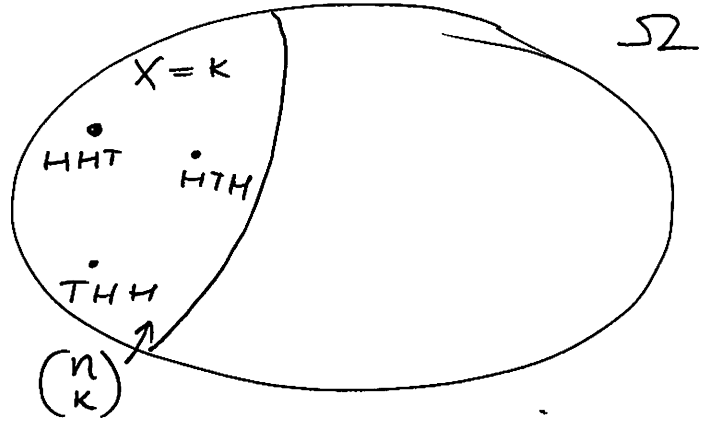
Average of a large number of trials.

$$P[H] = p$$

$$P[X=k]$$

$$= \binom{n}{k} p^k (1-p)^{n-k}.$$

$n=3 \quad k=2.$



Sample Space Ω :

$n = 3$.

$123 \to Y = 3$

$213,\ 321,\ 132 \to Y = 1$

$312,\ 231 \to Y = 0$

$6 = 3!$

Y is a function that maps Ω to $\mathbb{Z}$.

__Definition__

Integer random variable X is a function that maps Ω to $\mathbb{Z}$.

$$X : \Omega \to \mathbb{Z}.$$

__Events__: $X = k$ is an event.

$$(X = k) = \{\omega \in \Omega : X(\omega) = k\}$$

such that

Flip biased coin $P(H) = p$ until 1^{st} H's.

How long before 1^{st} H's.

W_1.

$$P[W_1 = n] = (1-p)^{n-1} \cdot p.$$

$$\begin{array}{c} \| \\ \{\underbrace{TTT\cdots T}_{n-1} H\} \end{array}$$

$$\sum_{n=1}^{\infty} (1-p)^{n-1} \cdot p = 1$$

$$p \sum_{k=0}^{\infty} (1-p)^{k}$$

$$= p \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1.$$

W_K = waiting time for K^{th} H's.

$$P[W_K = n] = \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$\underbrace{- - - - - - - - -}_{k-1 \text{ H's.}} \underline{H}$$

$$\sum_{n=k}^{\infty} P[W_k = n] = 1.$$

$$\sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k (1-p)^{n-k} = 1.$$

Flip biased coin n times

$S_n = \#$ H's.

$$P[S_n = k] = \binom{n}{k} p^k (1-p)^{n-k}.$$

$$\sum_{k=0}^{n} P[S_n = k] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$$

Binomial theorem.

$$1 = 1^n = [p + (1-p)]^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

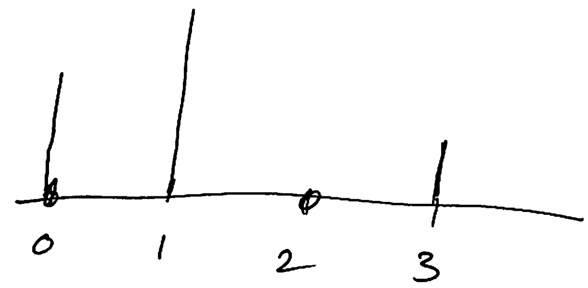
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

X $\quad P[X=0] = p_0 = \frac{2}{6}$

$P[X=1] = p_1 = \cancel{\frac{2}{6}}\ \frac{3}{6}$

$P[X=2] = p_2 = \frac{0}{6}$

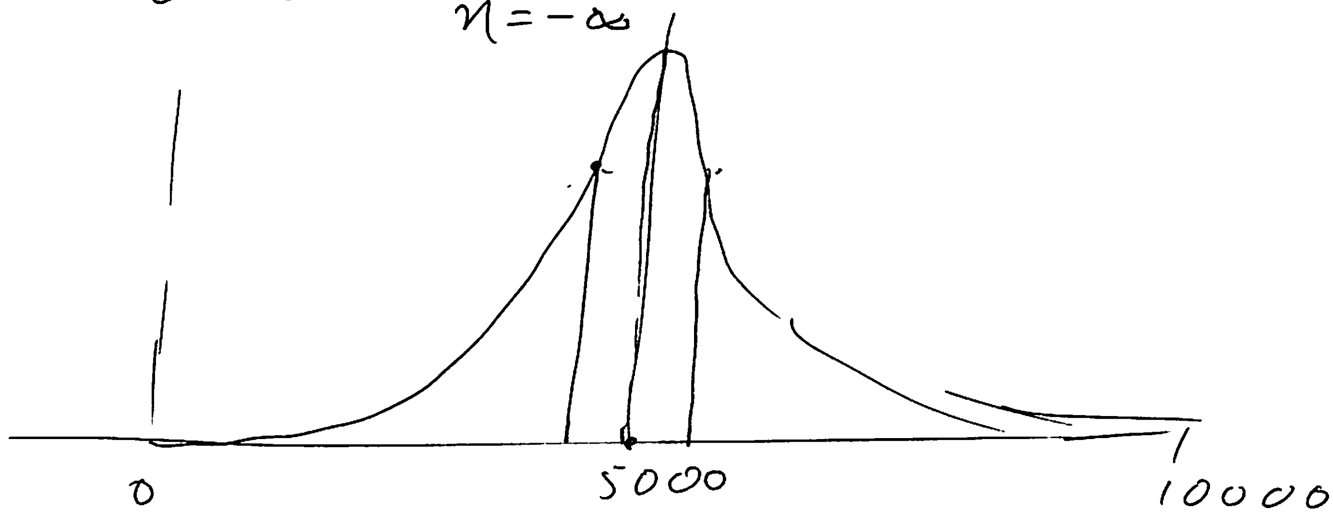
$P[X=3] = p_3 = \frac{1}{6}$



Prob Distribution of X

Expected value of X — $E[X]$.

$$E[X] = \sum_{n=-\infty}^{\infty} n \cdot P[X=n].$$



Envelopes example with $n = 3$.

$$E[X] = 0 \cdot \frac{2}{6} + 1 \times \frac{3}{6} + 2 \times \frac{0}{6} + 3 \times \frac{1}{6}.$$

$$= \frac{3}{6} + \frac{3}{6} = 1.$$

General n. - - - - - - - - - -

$P[X = k]$ hard!

$$E[X] = \sum_{k=0}^{n} k \cdot P[X = k]$$

Linearity of expectation:

Define random variable $X_i = \begin{cases} 1 & \text{if letter } i \text{ placed in envelope } i. \\ 0 & \text{o.w.} \end{cases}$

X_i is an indicator r.v.

$$X = X_1 + X_2 + \cdots + X_n.$$

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

$$\frac{16}{64} = \frac{1}{4}.$$

$$E[X_1] = 0 \times \left(1-\frac{1}{n}\right) + 1 \times \frac{1}{n}$$

$$= \frac{1}{n}.$$

$$P[X_1 = 1] = \frac{1}{n}.$$

$$P[X_1 = 0] = 1 - \frac{1}{n}.$$

$$E[X_i] = \frac{1}{n}.$$

$$E[X] = n \times \frac{1}{n} = 1.$$

Baseball cards:

$\boxed{\text{100 baseball cards.}}$

X = # distinct cards in 120 days.

E[# distinct cards you collect in ~~4 months~~ 120 days]

$X_i = \begin{cases} 1 & \text{if collect } ~~\text{cards}~~ \text{ } i^{th} \text{ player.} \\ 0 & \text{ow.} \end{cases}$

$P[X_B = 1] = E[X_B].$

$X = X_1 + X_2 + \cdots + X_{100}.$

$E[X] = 100\, E[X_1] = 100\, P[X_1 = 1] = 100\left[1 - \left(\frac{99}{100}\right)^{120}\right].$

$P[X_1 = 0] = \left[\frac{99}{100}\right]^{120}$

$P(X_1 = 1) = 1 - \left(\frac{99}{100}\right)^{120}$

$1, 2, 3, \cancel{4}, 5 \cancel{6}$.

n elts.
permute at random

Random permutation $\quad 3, 1, 2, 5, 4$

inversions
$= \{\{i,j\} : i<j$ but j before $i\}$.

$X =$ Count # inversions.

$E[X]$

$$X_{\{i,j\}} = \begin{cases} 1 & \text{if } i,j \text{ inverted} \\ 0 & \text{o/w.} \end{cases}$$

$$X = \sum_{\{i,j\}} X_{\{i,j\}}$$

$$P[X_{\{i,j\}} = 1] = \frac{1}{2}.$$

$$E[X_{\{i,j\}}] = 1/2$$

$$E[X] = \binom{n}{2} E[X_{\{1,2\}}] = \cancel{\tfrac{n}{2}} \times \cancel{\tfrac{1}{2}}.$$

$$= \frac{n(n-1)}{2} \times \frac{1}{2} = \frac{n(n-1)}{4}.$$

X, Y are r.v. on same sample space Ω

then $E[X+Y] = E[X] + E[Y]$.