CS70 Fall 2013
Lec 22

Expectation and Variance

2	1	3	6
4	0	5	9
4	6	0	10
10	7	8	25

$$E(X+Y) = E(X) + E(Y).$$

"# correct in 1st 2 envelopes."

	X	X_1	X_2
1 2 3	2	1	1
1 3 2	1	1	0
2 1 3		0	0
3 2 1	~~3~~ 1	0	1
3 1 2	0	0	0
2 3 1	0	0	0
	4/6	2/6	2/6

$$\boxed{X = X_1 + X_2}$$

$$E(X_1) = \sum_{i=0}^{1} i \, P[X_1 = i] = \sum_{\omega \in \Omega} X_1(\omega) \, P(\omega).$$

$(X_1 = 0)$

$(X_1 = 1)$

$$E(X+Y) = E(X) + E(Y).$$

$$\parallel$$

$$\sum_{\omega} P(\omega)\, \cancel{E}(X+Y)(\omega)$$

$$= \sum_{\omega} P(\omega)\left[X(\omega) + Y(\omega)\right].$$

$$= \sum_{\omega} P(\omega)\, X(\omega) + \sum_{\omega} P(\omega)\, Y(\omega).$$

$$= E(X) + E(Y).$$

$P[H] = p.$

W_1 = waiting time for first H's.

$P[W_1 = n] = (1-p)^{n-1} \cdot p.$

$\underbrace{TTTT\cdots T}_{n-1}H$

$$E(W_1) = \sum_{n=1}^{\infty} n \cdot (1-p)^{n-1} p = p \sum_{n=1}^{\infty} n(1-p)^{n-1}$$

$$= -p \sum_{n=1}^{\infty} \frac{d}{dp}(1-p)^n$$

$$= -p \frac{d}{dp} \sum_{n=1}^{\infty} (1-p)^n = -p(1-p)\sum_{n=0}^{\infty}(1-p)^n$$

$$= -p\frac{d}{dp}(1-p)\frac{1}{[1-(1-p)]}$$

$$= -p\frac{d}{dp}\frac{(1-p)}{p}$$

$$= \frac{1}{p}$$

$$E(W_1) = p \cdot 1 + (1-p)\left[1 + E(W_1)\right]$$

$$p\,E(W_1) = 1$$

$$E(W_1) = \frac{1}{p}.$$

W_k = Waiting time for k^{th} H's.

$$\cancel{E}P[W_k = n] = \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$\underbrace{-\ -\ -\ -\ -\ -\ -\ -\ -}_{k-1 \text{ H's.}}\ \underline{H}$$

$$\cancel{k}/p$$
$$\|$$

$$E(W_k) = \sum_{n=k}^{\infty} n\, P[W_k = n] = \sum_{n=k}^{\infty} n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$W_k = \cancel{W_1 +}\ \underline{G_1} + G_2 + \cdots + G_k.$$

$$W_1 = G_1$$
$$G_2$$

$$\begin{aligned} E(W_k) &= E(G_1 + \cdots + G_k) \\ &= E(G_1) + E(G_2) + \cdots + E(G_k). \\ &= k\,E(G_1) = k\,E(W_1) = \frac{k}{p}. \end{aligned}$$

$$\underbrace{\quad\_\_\_\_\_\_\ H}_{G_1 = W_1}\ \underbrace{\_\_\_\_\_\ H}_{G_2}\ \underbrace{\_\_\_\_\_\ H}_{G_3}\ \cdots$$

Variance

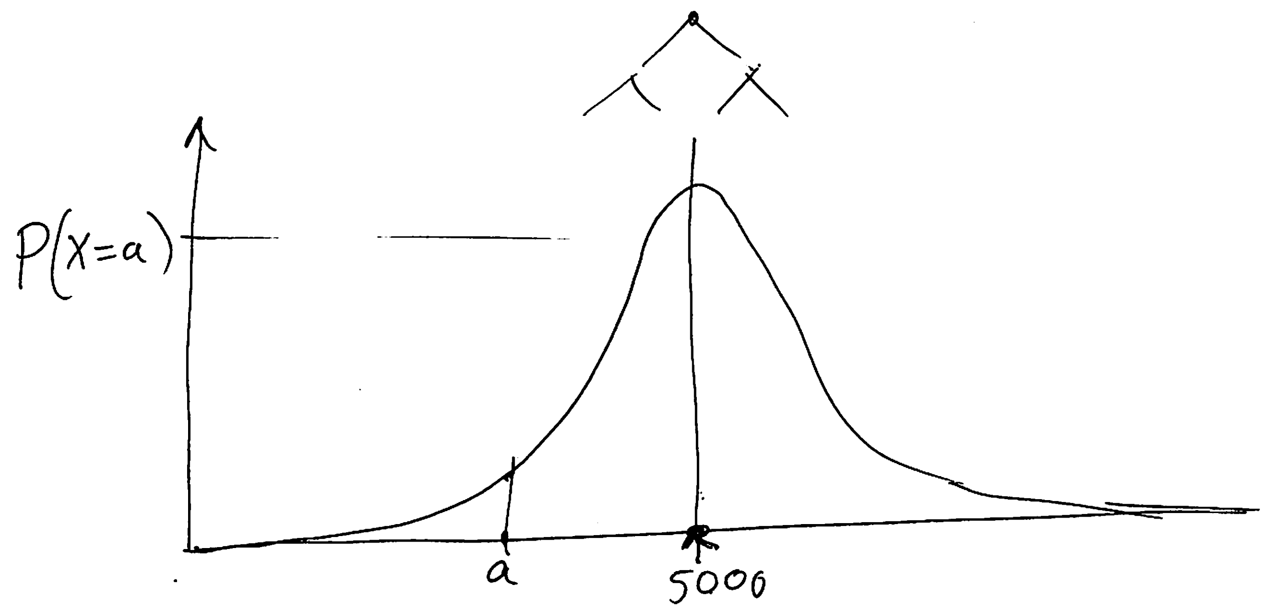
Flip a fair coin $n = 10{,}000$ times.

X = How many H's.

$X = X_1 + X_2 + \cdots + X_n$

$X_i = \begin{cases} 1 & \text{wp } \tfrac{1}{2} \\ 0 & \text{wp } \tfrac{1}{2} \end{cases}$

e.g. $= E(X) = 5000$



Y $\quad -5000 \quad 0 \quad 5000$

$\operatorname{Var}(X) = E\left[[X - 5000]^2\right] \to \frac{n}{4}$

Standard deviation $= \sigma(X) = \sqrt{\operatorname{Var}(X)} \to \frac{\sqrt{n}}{2}$.

Fact : $Var(X) = E(X^2) - (E(X))^2$

$(X-1)^2$		X	X^2
4	1 2 3	3	9
0	1 3 2, 3 2 1, 2 1 3,	1	1
1	3 1 2, 2 3 1	0	0

X = # letters in own envelope.

$$E(X) = \frac{1}{6}\times 3 + \frac{3}{6}\times 1 = 1$$

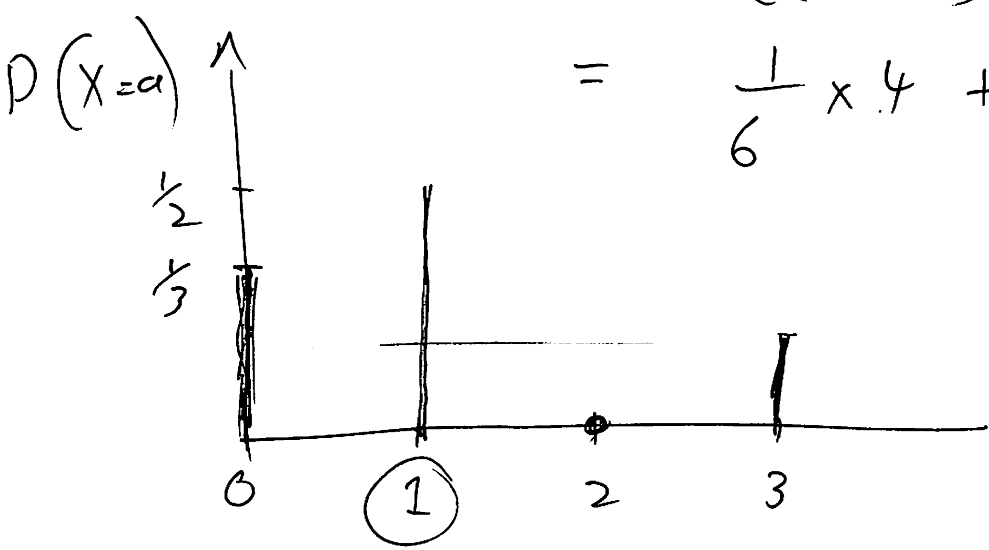
$$E(X^2) = \frac{1}{6}\times 9 + \frac{3}{6}\times 1 = \frac{9+3}{6} = 2$$

$$Var(X) = 2 - 1^2 = 1.$$

$$Var(X) = E\left((X - E(X))^2\right).$$

$$= E\left((X-1)^2\right).$$

$$= \frac{1}{6}\times 4 + \frac{2}{6}\times 1 = \frac{6}{6} = 1.$$



$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 \qquad E(X) = m$$

$$= E(X^2) - m^2$$

$$\mathrm{Var}(X) = E((X-m)^2).$$

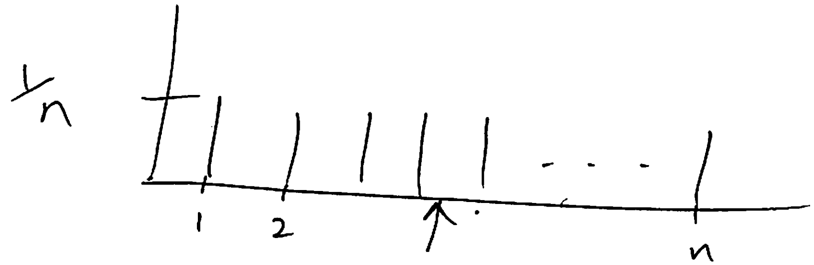
$$= E(X^2 - 2mX + m^2)$$

$$= E(X^2) + E(-2mX) + E(m^2)$$

$$= E(X^2) - 2m\,\underbrace{E(X)}_{m} + m^2$$

$$= E(X^2) - m^2.$$

Rolling dice $\underbrace{1, 2, 3, \ldots, n}_{P[\,]=\frac{1}{n}}$.



$$E(X) = \sum_{i=1}^{n} \frac{1}{n} \cdot i = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)}{2} = \frac{n+1}{2}.$$

$$Var(X) = \underline{\underline{E(X^2)}} - \left(\frac{n+1}{2}\right)^2.$$

$$E(X^2) = \sum_{i=1}^{n} \frac{1}{n} \cdot \underline{\underline{i^2}}$$

$$= \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6} \approx \frac{n^2}{3}.$$

$$Var(X) \approx \frac{n^2}{3} - \frac{n^2}{4} \approx \frac{n^2}{12}$$

$$\sigma(X) \approx \frac{n}{3.\ldots}$$

Independent Random Variables.

X, Y are indep r.v.

if $\forall a, b$ ~~P~~ $(X=a)$ & $(Y=b)$ are indep.

$$P\left[(X=a) \,\&\, (Y=b)\right] = P\left[X=a\right] \cdot P\left[Y=b\right].$$

X_1, X_2 are indep r.v.

$$\boxed{\text{If } X, Y \text{ are } \underline{\text{indep}} \text{ r.v's then } \operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y).}$$

$E(X+Y) = E(X) + E(Y)$ regardless of whether or not X, Y are indep.

Flip a fair coin n times. how many H's.

$$E[X] = \frac{n}{2} \qquad Var(X) = ? \qquad \cancel{p}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ flip H} \\ 0 & \text{o/w} \end{cases}$$

$$Var(X) = Var(X_1) + \cdots + Var(X_n)$$

$$= n\,Var(X_1)$$

$$= n\left[E[X_1^2] - E(X_1)^2\right]$$

$$= n\left[\frac{1}{2} - \left(\frac{1}{2}\right)^2\right]$$

$$= \frac{n}{4}.$$

$$\sigma(X) = \frac{\sqrt{n}}{2}$$

$$E[X_i] = \frac{1}{2}.$$

$$E[X_i^2] = \frac{1}{2}.$$

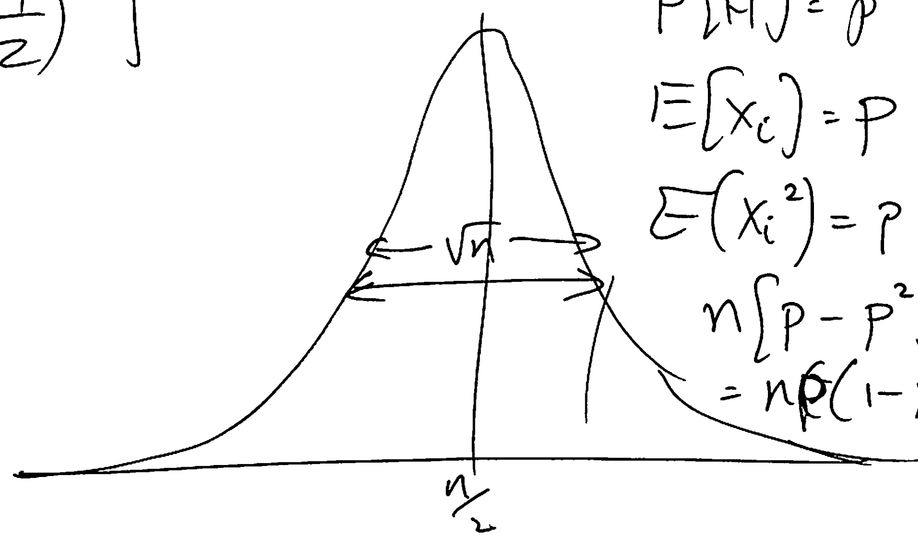
$$P[H] = p$$

$$E[X_i] = p$$

$$E(X_i^2) = p.$$

$$n[p - p^2]$$

$$= np(1-p)$$



If X, Y are indep r.v.

$$E[XY] = E[X] \cdot E[Y].$$

X, Y indep $\Rightarrow$

$$Var(X+Y) = Var(X) + Var(Y).$$

$$\|$$

$$E[(X+Y)^2] - [E(X+Y)]^2$$

$$= E[X^2 + 2XY + Y^2] - [E[X] + E(Y)]^2$$

$$= E[X^2] + \cancel{2E[XY]} + E[Y^2] - \left((E[X])^2 + \cancel{2E[X]E[Y]} + (E[Y])^2\right).$$

$$= \underbrace{\left[E(X^2) - (E(X))^2\right]}_{Var(X)} + \underbrace{\left(E(Y^2) - (E(Y))^2\right)}_{Var(Y)}.$$