CS70 Fall 2013
Lec 24

Some Important Distributions.

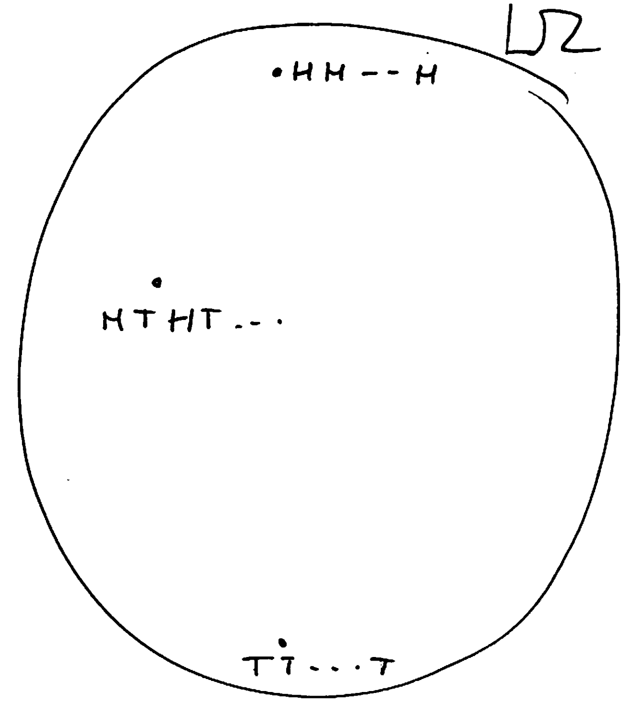
Binomial Distribution:

$Bin(n, p)$

Flipping a biased coin n times: $P[H] = p$.

S_n = # Heads.

$|\Omega| = 2^n$.



$$P[S_n = k] = \underline{\binom{n}{k}}\ \underline{p^k (1-p)^{n-k}}.$$

$$E[S_n] = np$$ linearity of expectation.

$$S_n = X_1 + X_2 + \cdots + X_n.$$

$$Var(S_n) = np(1-p).$$

$$= E(S_n^2) - (E(S_n))^2$$

$$= Var(X_1) + \cdots + Var(X_n) = n \cdot p(1-p).$$

X_i are indep.

$$E[X_1^2] - (E[X_1])^2$$

$$p - p^2 = p(1-p)$$

$$X_1 = \begin{cases} \text{if } H \text{ then } 1 \\ \text{if } T \text{ then } 0 \end{cases} \quad \begin{array}{|l} X_1^2 \\ 1 = 1^2 \\ 0 = 0^2 \end{array}$$

Geometric Distribution

Geom($\underline{\underline{p}}$).

Flip a coin ~~of~~ with $P[H] = p$ until first Heads.

W_1 = waiting time for 1st Heads.

$$P[W_1 = n] = (1-p)^{n-1} p$$

$$P[W_1 \geq n] = (1-p)^{n-1}$$

$$E[W_1] = \frac{1}{p}$$

$$\Vert$$

$$\sum_{n=1}^{\infty} n\, P[W_1 = n]$$

$$= \sum_{n=1}^{\infty} n p (1-p)^{n-1}$$

$$\sum_{n=1}^{\infty} p(1-p)^{n-1}$$

$$= p \sum_{m=0}^{\infty} (1-p)^m$$

$$= p \frac{1}{[1-(1-p)]} = 1.$$

$$P[W_1 \geq n] = (1-p)^{n-1}$$

$$E[W_1] = \sum_{i=1}^{\infty} P[W_1 \geq i]$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1}$$

$$= \sum_{j=0}^{\infty} (1-p)^{j}$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}.$$

Claim: ~~$E[X]$~~ X is non-neg integer r.v. then

$$E[X] = \sum_{i=1}^{\infty} i\, P[X = i] = \sum_{i=1}^{\infty} P[X \geq i].$$

Proof:

$$E[X] = \sum_{i=1}^{\infty} i\, P[X = i]. \qquad p_i = P[X = i].$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} i\, p_i \\
&= p_1 + 2p_2 + 3p_3 + 4p_4 + \cdots \\
&= p_1 + p_2 + p_3 + p_4 + \cdots \\
&\qquad\;\; + p_2 + p_3 + p_4 + \cdots \\
&\qquad\qquad\quad + p_3 + p_4 + \cdots \\
&\qquad\qquad\qquad\quad + p_4 \\
&= P[X \geq 1] + P[X \geq 2] + P[X \geq 3] + \cdots
\end{aligned}
$$

$$\sum_{i=1}^{\infty} P[X \geq i] = \sum_{i=1}^{\infty} \sum_{j \geq i} P[X = j] = \sum_{j=1}^{\infty} \Big(\sum_{i \leq j} P[X = j] \Big) = \sum_{j=1}^{\infty} P[X = j] \sum_{i \leq j} 1$$

$$= \sum_{j=1}^{\infty} j\, P[X = j].$$

Coupon Collector Problem:

n players.

How many boxes of cereal must you consume? (# boxes)

X = waiting time till see all cards.

$$X = X_1 + X_2 + \cdots + X_n.$$

example: $n = 3$.

B B A B A A C

$X_1 = 1 \quad X_2 = 2 \quad X_3 = 4.$

$X_1 + X_2 + X_3 = 7$

X_i = # boxes after got $i-1$ different cards till you got i^{th} type.

$$E[X] = E[X_1] + \cdots + E[X_n].$$

$$E[X_i] = ?$$

$E[X_1] = 1$

$E[X_2] = \frac{n}{n-1} \qquad P[H] = \frac{n-1}{n}.$

$E[X_3] = \frac{n}{n-2}. \qquad P[H] = \frac{n-2}{n}.$

$$E[X] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$$

$$= n\left[\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1}\right] = n\underbrace{H_n}_{n^{th}\text{ Harmonic number}} \approx n \ln n.$$

Poisson Distribution

Poisson (λ). $\qquad\qquad \lambda = 2.$

$$P[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}$$

$$P[X=0] = \frac{\lambda^0}{0!} e^{-\lambda} = \frac{1 \cdot e^{-2}}{1} = e^{-2}.$$

$$P[X=2] = \frac{2\cancel{2}^2}{2!} e^{-2}. = \left(\frac{2}{e}\right)^2 \frac{1}{2}.$$

"Rare events"

Crossovers in chromosomes.
Mutations in DNA sequences.
occurrences of disease.
births.
cell phone calls placed $\longrightarrow \lambda = 10.$

$$\sum_{i=0}^{\infty} P[X=i] = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= e^{-\lambda}\left(\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}\right) = e^{-\lambda} \cdot e^{\lambda} = 1.$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$E[X] = \sum_{i=0}^{\infty} i\, P[X=i] = \sum_{i=0}^{\infty} i\, \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} e^{-\lambda}$$

$$= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda}$$

$$= \underline{\underline{\lambda}}.$$

Poisson: "rare events"

$\mathrm{Var}(X) = np(1-p)$
$= \lambda(1 - \cancel{\lambda/n})$
$= \lambda$

Flip a coin n times: $P[H] = p$.

$E[\#\text{Heads}] = np = \lambda = \underline{\underline{2}}$ $\quad n \to \infty$

$p = \frac{\lambda}{n}$.

$$p_i = P[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}.$$

$$p_0 = (1-p)^n = \left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$$

$\left(1 - \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} = e^{-1}$

$\left(1 - \frac{\lambda}{n}\right)^{n/\lambda} \to \frac{1}{e}$

$\left(1 - \lambda/n\right)^{n/\lambda \cdot \lambda} \to e^{-\lambda}$.

$$\frac{p_i}{p_{i-1}} = \frac{\binom{n}{i} p^i (1-p)^{n-i}}{\binom{n}{i-1} p^{i-1} (1-p)^{n-i+1}}$$

$$= \frac{\frac{\cancel{n!}}{i!\,\cancel{(n-i)!}} \left(\frac{\lambda}{n}\right)^i \cancel{\left(1 - \frac{\lambda}{n}\right)^{n-i}}}{\frac{\cancel{n!}}{(i-1)!\,(n-i+1)!} \cancel{\left(\frac{\lambda}{n}\right)^{i-1}} \left(1 - \lambda/n\right)^{\cancel{n-i+1}}} = \frac{(n-i+1)\lambda}{i \; n\left(1 - \frac{\lambda}{n}\right)}$$

$$\xrightarrow[n\to\infty]{} \frac{\lambda}{i}$$

By induction on i:

$$p_{i-1} = \frac{\lambda^{i-1}}{(i-1)!} e^{-\lambda} \Rightarrow p_i = \frac{\lambda}{i} \cdot \frac{\lambda^{i-1}}{(i-1)!} e^{-\lambda}$$