Infinity, Uncountable, Uncomputable

Alan Turing: Test whether P halts on every input.

Turing

Uncomputability

Diagonalization

Cantor Infinity

Uncountable

Proofs

Gödel

Self Reference:

This statement is false. Paradox.

Virus

$f: \{1, 2, 3\} \longrightarrow \{A, B, C\}$.

bijection f.

$\mathbb{N}$ natural numbers — countably infinite.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

Hotel $\mathbb{Z}^+$:

0 check in.

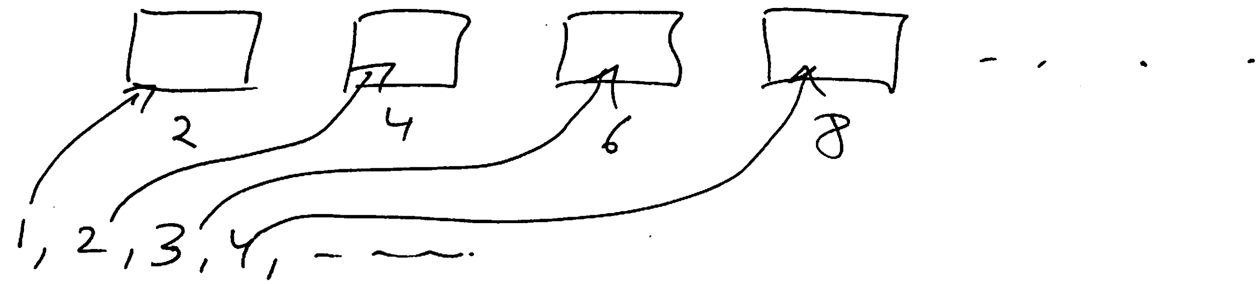
Bijection $f: \mathbb{N} \to \mathbb{Z}^+$

$f(x) = x + 1$

1-1 $\quad \underset{x+1 \,=\, y+1}{f(x) = f(y)} \Rightarrow x = y$.

onto $\longrightarrow$

Hotel $2\mathbb{Z}^+$



$f(x) = : \mathbb{Z}^+ \to 2\mathbb{Z}^+$

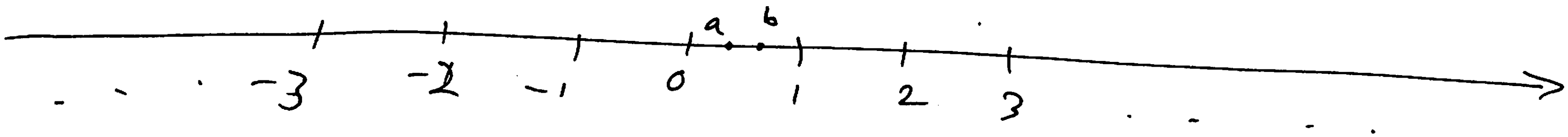
$f(x) = 2x$.

$\mathbb{N}$ vs $\mathbb{Z}$

<u>exercise</u>: find a bijection.

$0, 1, 2, 3, 4, 5 - - - -$

$$\underbrace{\cdots -3, -2, -1,}_{\text{odd natural numbers}} \underbrace{0, 1, 2, 3, 4, 5 \ldots}_{\text{even}}$$



Rationals $\mathbb{Q}$.

Rationals are countable.

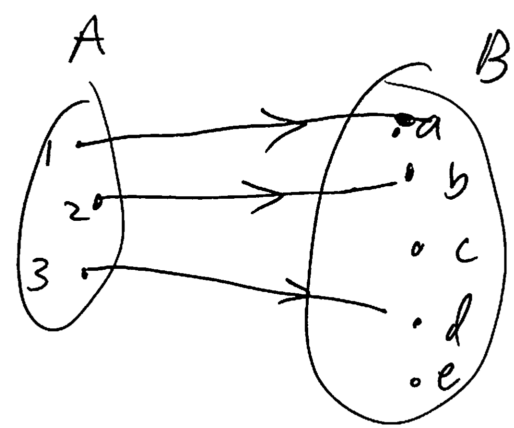
bijection $f : \mathbb{N} \to \mathbb{Q}$.

In spirit:
to show $a = b$
$\iff$ $a \leq b$
and $b \leq a$.

Cantor - Bernstein Theorem:

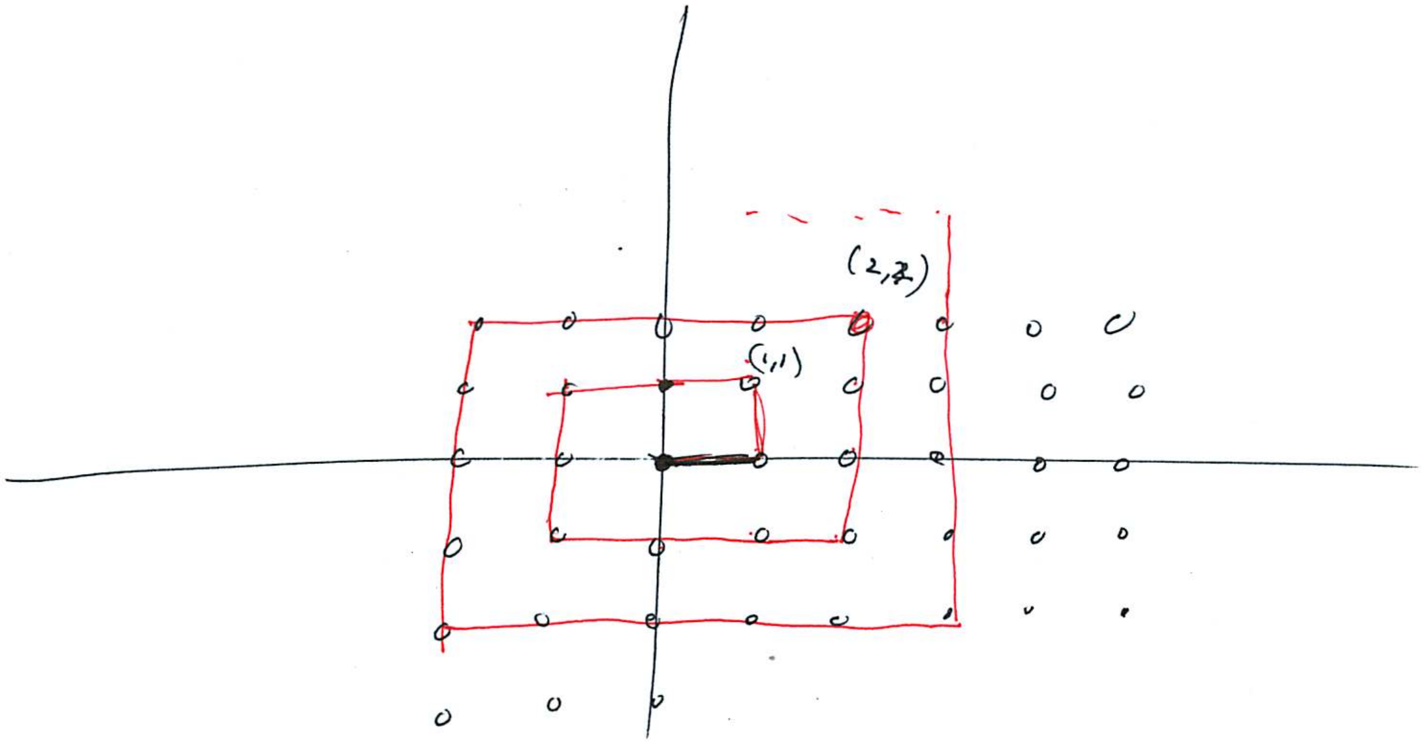
If there is a 1-1 function $f : A \to B$ and
a 1-1 function $g : B \to A$
then $\exists$ bijection $h : A \to B$.

f:



$f : \mathbb{Q} \longrightarrow \mathbb{N}$ f is $1-1$.

$x \in \mathbb{Q}$ then $x = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ $b \neq 0$.



$f\left(\frac{2}{2}\right) = \underline{\underline{13}}$

$f\left(\frac{1}{1}\right) =$

Rational number $\overset{\overset{+1,-1}{\|}}{s}\ \dfrac{a}{b} \longrightarrow 2^a\, 3^b\, 5^{\frac{s+1}{2}}$

$$\frac{2}{3} \longrightarrow \underline{\underline{2^2\, 3^3}} = 4 \times 27$$

$$= \underbrace{108}_{= 2^2 3^3}$$

$p(x) = ax + b \qquad a, b \in \mathbb{Z}$

$S = \{ p(x) : ax = b \quad a, b \in \mathbb{Z} \}.$

S is countable.

Set of all finite ~~binary~~ ternary strings is countable.

1 0 2 1 2 2 0 0 1

$$S = \{0,1,2\}^* = \{\underset{\substack{\uparrow \\ \text{empty string}}}{\varepsilon}, 0, 1, 2, 00, 01, 02, 10, 11, 12, 20, 21, 22, \ldots\}.$$

lexicographic order

$$\frac{a}{b} \longrightarrow \underbrace{\qquad}_{a \text{ in binary}} 2 \underbrace{\qquad}_{b \text{ in binary}}.$$

$$\frac{2}{5} \longrightarrow 102101$$

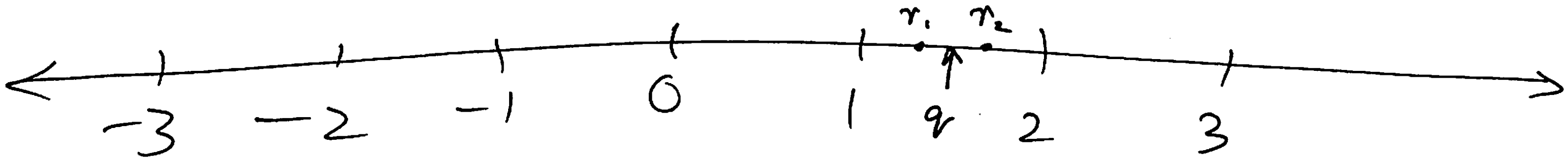
$$T = \{\text{all polynomials with integer coefficients}\}.$$

$$a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \qquad a_i \in \mathbb{Z},\ d \in \mathbb{N}.$$

$$\Downarrow$$

$$\underbrace{\qquad}_{a_d \text{ in binary}} 2 \underbrace{\qquad}_{\substack{a_{d-1} \\ \text{in binary}}} 2 \underbrace{\qquad}_{a_{d-2}} 2 \cdots 2 \underbrace{\qquad}_{\substack{a_0 \\ \text{in binary}}}.$$

Reals countable ?



Rationals are countable.

Rationals are dense.

$$r_1 = 2.3578012\ldots\ldots$$

$$r_2 = 2.3578102\ldots\ldots$$

$$q = 2.357805 = \frac{2357805}{1000000}$$

Theorem: Reals [0,1] are uncountable.

Outline of Proof: Assume for contradiction that the reals are countable.

$\Rightarrow$ bijection.

List the real numbers in order.

~~Find~~ ~~Demonstrate~~ Exhibit a real number that is not on the list.

Contradiction $\times$

Proof:

0.5 1 2 8 3 2 . . . . .

0.1 2 3 4 5 6 - - -

0.0 0 0 1 0 0 0 1 0 1 0 1 - - .

0.2 3 4 5 1 2 3 4

0.7 2 1 7 2 1 . . .

0.1 4 1 5 2 9 - - . .

.~~5 1 2 8~~

r' .5 2 0 5 2 9 - - - - - - -

r 7 4 2 7 4 1 2 ~~5~~ 9 → 3

Diagonal ⟷ real number
$r' = .520529....$

Add 2 mod 10 $r = .742741259...$

Where ~~does~~ is r in the list?

$.2\underline{3}999999999.... = .2\underline{4}$

$R = A \cup T$

Algebraic — transcendental π, e

$\sqrt{2}$ $\quad x^2 - 2 = 0$

Cardinality of A — countable.

<u>Theorem</u>: A, B are countable then so is $A \cup B$.

$A \to$ even $\mathbb{N}$

$B \to$ odd $\mathbb{N}$.

$A \longleftrightarrow 0, 1, 2, 3, 4, \ldots$

$B \longleftrightarrow 0, 1, 2, 3, 4$

T is uncountable — proof by contradiction.