1. Find a polynomial of degree at most 2 which passes through the points \((1, 2), (2, 5), (3, 12)\). Use linear equations as opposed to Lagrangian interpolation. We use the linear equations below to solve for \(a, b\) and \(c\):

\[
\begin{align*}
    a_2 + a_1 + a_0 &= 2 \\
    4a_2 + 2a_1 + a_0 &= 5 \\
    9a_2 + 3a_1 + a_0 &= 12
\end{align*}
\]

We obtain the first equation by plugging in 1 to \(a_2x^2 + a_1x + a_0\) and setting it equal to 2, as given by the first point \((1, 2)\). We use a similar method to obtain the last two equations. After solving, we obtain \(a_2 = 2, a_1 = -3\) and \(a_0 = 3\), giving the polynomial \(2x^2 - 3x + 3\).

2. When working in \(GF(11)\), we want to send a message \((5, 2)\). The message we send might get corrupted at \(k = 1\) place.

   (a) First find a polynomial \(P(x)\) such that \(P(1) = 5, P(2) = 2\). Using the same technique as above, we obtain the degree 1 polynomial \(P(x) = 8x + 8\).

   (b) You should be sending more than two characters to ensure that you can recover from 1 general error. How many should you send? You need to send \(n + 2k\) characters, where \(n\) is the length of the original message (2) and \(k = 1\), so you need to send 4 characters.

   (c) Assuming the answer to the previous question was \(m\), find out the actual message you’ll be sending by evaluating \(P(x)\) at \(x = 1, 2, \ldots, m\). We’ll evaluate \(P(x)\) at \(x = 1, 2, 3, 4\). \(P(1)\) and \(P(2)\) are given above. \(P(3) = 10\) and \(P(4) = 7\). The actual message will be 5, 2, 10, 7.

   (d) Assume that the message you send gets corrupted at the 2nd character. Increase the value you computed by 1 to get the corrupted character. Remember that the person receiving the message does not know that it was the 2nd character that got corrupted. What will the error-locating polynomial be here? The error locating polynomial is \(E(x) = x - 2\).

   (e) To decode the message you set-up the polynomial \(Q(x)\) as \(P(x)E(x)\) where \(P\) is the original polynomial and \(E\) is the error-locating polynomial. Remember that the receiver does not know what any of these polynomials are yet. What is the degree of \(Q\)? The degree of \(Q(x)\) is \(n + k - 1\). In this example, \(n = 2\) and \(k = 1\), so the degree is 2.

   (f) Remember that the equation \(Q(x) = r_xE(x)\) is satisfied where \(x = 1, \ldots, m\) and \(r_x\) is the \(x\)-th character received. Why is this true? Give a short justification. If no error occurs, then since \(Q(x) = P(x)E(x)\) and \(P(x) = r_x\), the equation is satisfied. If an error occurs, then \(Q(x) = 0\) as does \(E(x)\).
(g) Now the receiver writes $Q(x)$ and $E(x)$ in the most general format possible (i.e. with arbitrary coefficients). Write $Q(x)$ and $E(x)$ replacing their coefficients with variables. Remember that even though the receiver knows nothing about the message yet, he knows one of the coefficients of $E(x)$. What is that coefficient? Let $Q(x) = a_2 x^2 + a_1 x + a_0$ and $E(x) = x + b_0$—the receiver knows that the coefficient of $x$ is 1.

(h) Now write down the system of linear equations corresponding to $Q(x) = r E(x)$. The variables are the coefficients of $Q$ and $E$. The first equation will be $a_2 + a_1 + a_0 = 5(1 + b_0)$, which simplifies to $a_2 + a_1 + a_0 + 6b_0 = 5$. We can repeat this process for $x = 2, 3, 4$ to obtain the following equations:

\[
\begin{align*}
    a_2 + a_1 + a_0 + 6b_0 &= 5 \\
    4a_2 + 2a_1 + a_0 + 8b_0 &= 6 \\
    9a_2 + 3a_1 + a_0 + b_0 &= 8 \\
    5a_2 + 4a_1 + a_0 + 4b_0 &= 6
\end{align*}
\]

(i) Solve the linear system. Did you get the error-locating polynomial you expected? After solving, we obtain $b_0 = -2$ and $a_2 = 8, a_1 = 3, a_0 = 6$. Hence $Q(x) = 8x^2 + 3x + 6$ and $E(x) = x - 2$.

(j) How would one go from knowing $E$ and $Q$ to finding $P$? Since $Q(x) = P(x) E(x)$, we can compute $P(x) = \frac{Q(x)}{E(x)}$.

3. What happens in the error-correcting method if there are actually no errors? Try the previous problem but don’t corrupt the 2nd character. Now our system of equations is:

\[
\begin{align*}
    a_2 + a_1 + a_0 + 6b_0 &= 5 \\
    4a_2 + 2a_1 + a_0 + 9b_0 &= 4 \\
    9a_2 + 3a_1 + a_0 + b_0 &= 8 \\
    5a_2 + 4a_1 + a_0 + 4b_0 &= 6
\end{align*}
\]

After solving, you can see that we have a degenerate system of linear equations, and $b_0$ can be any value.