
1. Finish Polynomials and Secrets.
2. Finite Fields: Abstract Algebra
3. Erasure Coding
Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact:
There is exactly 1 polynomial of degree \( \leq d \) with arithmetic modulo prime \( p \) that contains \( d + 1 \) pts.
Note: The points have to have different \( x \) values!

Shamir's \( k \) out of \( n \) Scheme:

1. Choose \( a_0 = s \), and random \( a_1, \ldots, a_{k-1} \).
2. Let \( P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0 \) with \( a_0 = s \).
3. Share \( i \) is point \((i, P(i) \mod p)\).

Robustness: Any \( k \) shares gives secret.
Knowing \( k \) pts, find unique \( P(x) \), evaluate \( P(0) \).

Secrecy: Any \( k - 1 \) shares give nothing.
Knowing \( \leq k - 1 \) pts, any \( P(0) \) is possible.
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There exists a polynomial...
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**Proof of at least one polynomial:**
Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$. 

\[ \Delta_i(x) = \prod_{j \neq i} (x - x_j) \cdot \prod_{j \neq i} (x_i - x_j). \]

The numerator is $0$ at $x_j \neq x_i$. The denominator makes it $1$ at $x_i$.

\[ P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_{d+1} \Delta_{d+1}(x). \]

This construction proves the existence of a polynomial!
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Degree 1 polynomial, \( P(x) \), that contains \((1, 3)\) and \((3, 4)\)?
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Work modulo 5.
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Put the delta functions together.
Simultaneous Equations Method.

For a line, \( a_1 x + a_0 = mx + b \) contains points \((1, 3)\) and \((2, 4)\).
Simultaneous Equations Method.

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\[
P(1) =
\]

\[
\]

Subtract first from second.

\[
m + b \equiv 3 \pmod{5}
\]

\[
m \equiv 1 \pmod{5}
\]

Backsolve:

\[
b \equiv 2 \pmod{5}
\]

Secret is 2.

And the line is...

\[
x + 2 \mod{5}
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For a line, \( a_1 x + a_0 = mx + b \) contains points \((1, 3)\) and \((2, 4)\).

\[
P(1) = m(1) + b \equiv m + b
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Subtract first from second.

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Subtract first from second..

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m \equiv 1 \pmod{5} \\
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And the line is\( x + 2 \pmod{5} \).
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And the line is...

\[ x + 2 \mod 5. \]
For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2); (2,4); (3,0)$.

Plug in points to find equations.

$$P(1) = a_2 + a_1 + a_0 \equiv 2 \pmod{5}$$

$$P(2) = 4a_2 + 2a_1 + a_0 \equiv 4 \pmod{5}$$

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Subtracting the 2nd from the 3rd yields:

$$a_1 = 1.$$  

$$a_0 = (2 - 4a_1)^2 - 1 = (-2)^2(2 - 1) = 2.$$  

$$a_2 = 2 - 1 - 4a_0 = 2 - 1 - 4(-2) = 9 \equiv 4 \pmod{5}.$$  

So polynomial is $2x^2 + x + 4 \pmod{5}$. 
For a quadratic polynomial, \( a_2 x^2 + a_1 x + a_0 \) hits (1,2); (2,4); (3,0). Plug in points to find equations.
Quadratic

For a quadratic polynomial, \( a_2 x^2 + a_1 x + a_0 \) hits \((1,2);(2,4);(3,0)\). Plug in points to find equations.

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In general..

Given points: \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).
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  a_{k-1}x_1^{k-1} + \cdots + a_0 & \equiv y_1 \pmod{p} \\
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Will this always work?
In general.

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As long as solution exists and it is unique! And...
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\[
\vdots \quad \vdots \quad \vdots 
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Will this always work?

As long as solution \textbf{exists} and it is \textbf{unique}! And...

\textbf{Modular Arithmetic Fact:} Exactly 1 polynomial of degree \(\leq d\) with arithmetic modulo prime \(p\) contains \(d + 1\) pts.
Uniqueness.

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Existence: Lagrange interpolation. Uniqueness?

**Uniqueness Fact.** At most one degree $d$ polynomial hits $d + 1$ points.
**Uniqueness Fact.** At most one degree $d$ polynomial contains $d + 1$ points.
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**Proof:**
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**Proof:**

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.
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**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.

Assume two different polynomials $Q(x)$ and $P(x)$ hits $d + 1$ points.
Uniqueness Fact. At most one degree $d$ polynomial contains $d + 1$ points.

Proof:

Roots fact: Any degree $d$ polynomial has at most $d$ roots.
Assume two different polynomials $Q(x)$ and $P(x)$ hits $d + 1$ points. $R(x) = Q(x) - P(x)$ has $d + 1$ roots and is degree $d$. 
Uniqueness Fact. At most one degree $d$ polynomial contains $d + 1$ points.

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Contradiction.
Uniqueness Fact. At most one degree $d$ polynomial contains $d + 1$ points.

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**Contradiction.**

Must prove **Roots fact.**
Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.
Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

$$
\begin{array}{cccc}
\phantom{-}4 & x & & \\
\hline
x & - & 3 & ) \ 4x^2 & - & 3x & + & 2 \\
\phantom{-}4x^2 & - & 3x & + & 2 \\
\hline
& & 4x & + & 2 \\
\phantom{-}4x & - & 2 & \\
\hline
& & 4 &
\end{array}
$$

$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}$
Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c|cc}
& 4x & \\
\hline
x - 3 & 4x^2 & -3x & +2 \\
\hline
& 4x^2 & -2x \\
\hline
& 4x & +2 \\
\hline
& 4x & -2 \\
\hline
& 4 & +2 \\
\end{array}
\]

$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4$ (mod 5)

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x - a)Q(x) + r$

$r$ is degree 0 polynomial, or a constant!
Polynomial Division.

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\hline
x & - & 3 & ) \\ 4x^2 & - & 3x & + & 2 \\
\downarrow & & & & \\
4x^2 & - & 2x & & \\
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\[
\begin{array}{ccccc}
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\hline
x - 3 ) & 4x^2 & - & 3x & + & 2 \\
 & 4x^2 & - & 2x \\
\hline
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\hline
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\[
\begin{array}{c|cc}
\text{} & 4x^2 & - 2x \\
\hline
\text{--} & \text{--} & \text{--} \\
\end{array}
\]

\[
\begin{array}{c|cc}
\text{} & 4x & + 2 \\
\hline
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\[
\begin{array}{c|cc}
 & 4x & + 4 \\
\hline
x - 3 & 4x^2 & - 3x & + 2 \\
& 4x^2 & - 2x & \\
\hline
& 4x & + 2 & \\
& 4x & - 2 & \\
\hline
& 4 & \\
\end{array}
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Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c c c}
4x^2 & - & 3x & + & 2 \\
\hline
x - 3 & ) & 4x^2 & - & 3x & + & 2 \\
 & & 4x^2 & - & 2x \\
 & & \hline
 & & 4x & + & 2 \\
 & & 4x & - & 2 \\
 & & \hline
 & & 4 \\
\end{array}
\]

$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}$
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\[ \frac{4x^2 - 3x + 2}{x - 3} = 4x + 4 + \frac{4}{x - 3} \]

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4 & x & + 4 & r & 4 \\
\hline
x & - & 3 & ) & 4x^2 & - 3x & + 2 \\
& - & 4x^2 & - 2x & & & \\
\hline
4x & + & 2 & & & \\
& - & 4x & - 2 & & \\
\hline
& & 4 & & & 
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\hline
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\quad 4x^2 - 2x \\
\hline
\quad 4x + 2 \\
\quad 4x - 2 \\
\hline
\quad 4
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Uniqueness:
- At most \( d \) roots for degree \( d \) polynomial.
Finite Fields

Proof works for reals, rationals, and complex numbers.
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Arithmetic modulo a prime $p$ is a **finite field** denoted by $F_p$ or $GF(p)$.
Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.
Secret Sharing

**Modular Arithmetic Fact:** Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d + 1$ points.
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Need $p > n$ to hand out $n$ shares: $P(1), \ldots, P(n)$.

For $b$-bit secret, must choose a prime $p > 2^b$.

Theorem: There is always a prime between $n$ and $2^n$.

Working over numbers within 1 bit of secret size.

Minimal!

With $k$ shares, reconstruct polynomial, $P(x)$.

With $k - 1$ shares, any of $p$ values possible for $P(0)$!

(Within 1 bit of) any $b$-bit string possible!

(Within 1 bit of) $b$-bits are missing: one $P(i)$.

Within 1 of optimal number of bits.
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**Minimal!**
Efficiency.

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For $b$-bit secret, must choose a prime $p > 2^b$.

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(Within 1 bit of) any $b$-bit string possible!
(Within 1 bit of) $b$-bits are missing: one $P(i)$.
Within 1 of optimal number of bits.
Runtime.

1. Evaluate degree $n - 1$ polynomial $n + k$ times using $\log p$-bit numbers. $O(kn \log 2^p)$.

2. Reconstruct secret by solving system of $n$ equations using $\log p$-bit arithmetic. $O(n^3 \log 2^p)$.

3. Matrix has special form so $O(n \log n \log 2^p)$ reconstruction.

Faster versions in practice are almost as efficient.
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Runtime: polynomial in $k$, $n$, and $\log p$.

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A bit of counting.

What is the number of degree $d$ polynomials over $GF(m)$?
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- \( m^{d+1} \): \( d + 1 \) coefficients from \( \{0, \ldots, m - 1\} \).
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- $m^{d+1}$: $d + 1$ coefficients from $\{0, \ldots, m-1\}$.
- $m^{d+1}$: $d + 1$ points with $y$-values from $\{0, \ldots, m-1\}$
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Infinite number for reals, rationals, complex numbers!
Erasure Codes.

Satellite

GPS device
Erasure Codes.

Satellite

3 packet message.

GPS device
Erasure Codes.

Satellite 3 packet message.

GPS device Lose 3 out 6 packets.
Erasure Codes.

3 packet message. So send 6!

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Gets packets 1, 1, and 3.
Problem: Want to send a message with $n$ packets.
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**Question:** Can you send \( n + k \) packets and recover message?
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Channel: Lossy channel: loses $k$ packets.
Question: Can you send $n + k$ packets and recover message?
On Friday!