Lecture 14. Outline.

1. Finish Polynomials and Secrets.
2. Finite Fields: Abstract Algebra
3. Erasure Coding

Modular Arithmetic Fact and Secrets

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Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

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$$P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_{d+1} \Delta_{d+1}(x).$$

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Construction proves the existence of a polynomial!

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Put the delta functions together.

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And the line is...

$$x + 2 \mod 5.$$

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So polynomial is $2x^2 + 1x + 4 \pmod 5$

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Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

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$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree d.

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Plugin a: $P(a) = r = 0$. $\square$

Lemma 2: $P(x)$ has d roots; $r_1, \ldots, r_d$ then
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Proof Sketch: By induction.

Base Case: degree 0. No roots.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1.
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Roots fact: Any degree d polynomial has at most d roots.

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Faster versions in practice are almost as efficient.

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Infinite number for reals, rationals, complex numbers!

Erasure Codes.

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GPS device

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3 packet message.

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Lose 3 out 6 packets.

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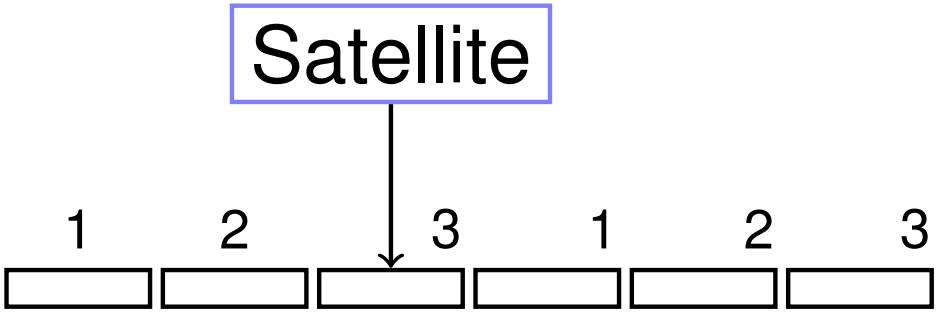

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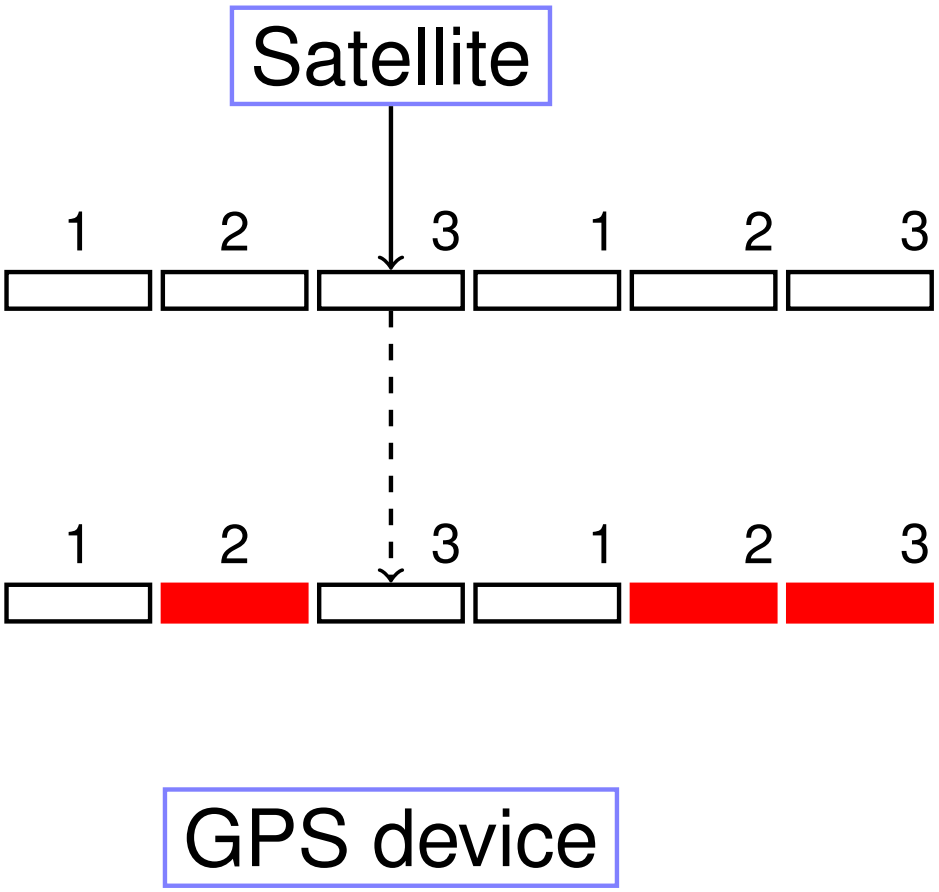

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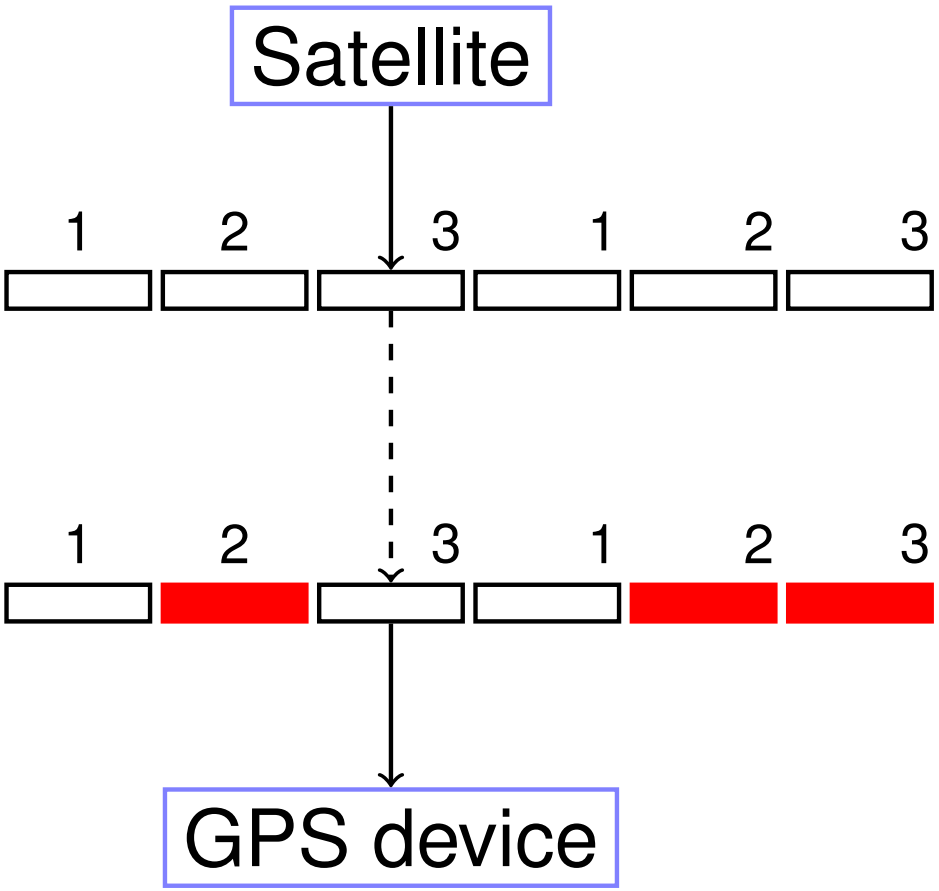

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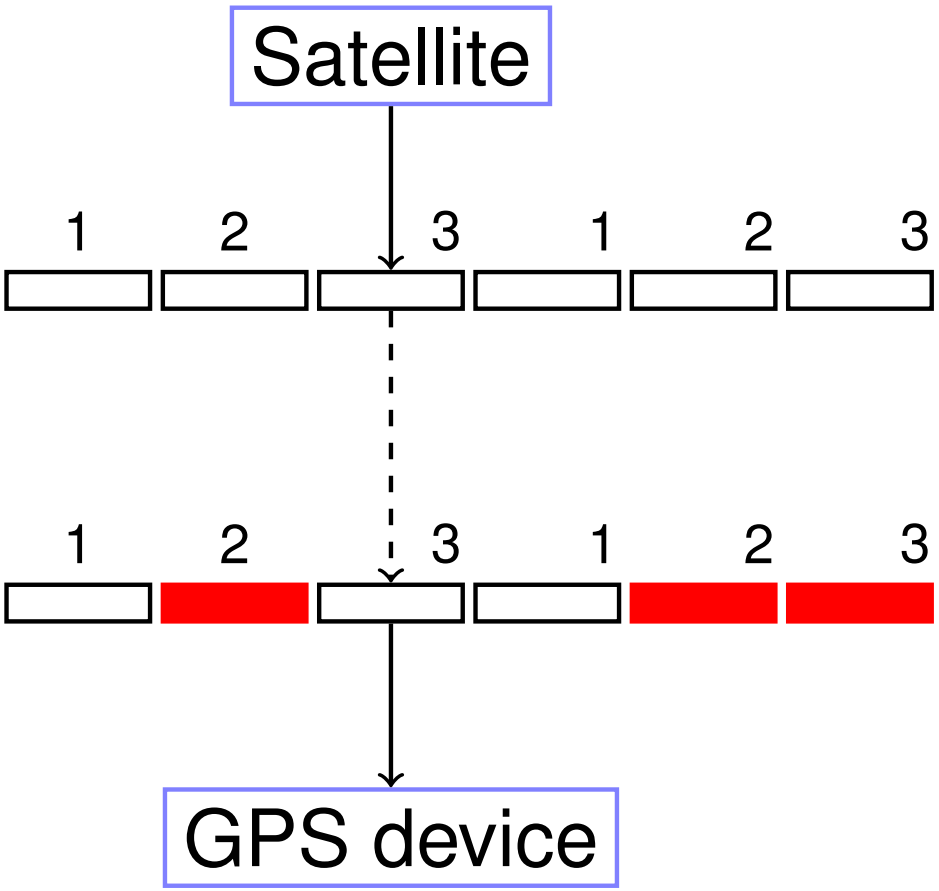

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Gets packets 1,1,and 3.

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On Friday!