Brief Comment.
Math and Computer Science Theory.
E.g. There is a multiplicative inverse modulo a prime.
Existence: pigeonhole principle and divisibility argument.
Does not give efficient method to find inverse.
Extended GCD gives inverse.
and proof of existence.
E.g. There is a way to recover from $k$ errors in $n + 2k$ packets.
Unique reconstruction that is consistent with $n + k$ points.
Does not give efficient method to reconstruct.
Berlekamp-Welsh: Error Locator polynomials!
Efficient reconstruction.
It's all math, done by mathematicians!
Efficient methods to construct objects important in CS.

What's to come? Probability.
A bag contains:

![Colorful dots diagram]

What is the chance that a ball taken from the bag is blue?
Today: Counting!

Count?
How many outcomes possible for $k$ coin tosses?
How many poker hands?
How many handshakes for $n$ people?
How many diagonals in a convex polygon?
How many 10 digit numbers?
How many 10 digit numbers without repetition?

Using a tree..
How many 3-bit strings?
How many different sequences of three bits from {0,1}?
How would you make one sequence?
How many different ways to do that making?
8 leaves which is $2 \times 2 \times 2$. One leaf for each string.
8 3-bit strings!
First Rule of Counting: Product Rule

Objects made by choosing from $n_1$, then $n_2$, ..., then $n_k$ the number of objects is $n_1 \times n_2 \times \cdots \times n_k$.

In picture, $2 \times 2 \times 3 = 12!$

Using the first rule..

How many outcomes possible for $k$ coin tosses?
2 ways for first choice, 2 ways for second choice, ...

2 ways for second choice, ...

$2 \times 2 \cdots \times 2 = 2^k$

How many 10 digit numbers?
10 ways for first choice, 10 ways for second choice, ...

10 ways for second, ...

$10 \times 10 \cdots \times 10 = 10^k$

How many $n$ digit base $m$ numbers?
m ways for first, $m$ ways for second, ...

$m^n$

Permutations.

How many 10 digit numbers without repeating a digit?
10 ways for first, 9 ways for second, 8 ways for third, ...

$10 \times 9 \times 8 \cdot \cdot \cdot = 10!$.

How many different samples of size $k$ from $n$ numbers without replacement.

$n$ ways for first choice, $n-1$ ways for second, $n-2$ choices for third, ...

$\ldots \cdot (n-1) \cdot (n-2) \cdot (n-k) = \frac{n!}{(n-k)!}$.

How many orderings of $n$ objects are there?

Permutations of $n$ objects.

$n$ ways for first, $n-1$ ways for second, $n-2$ ways for third, ...

$\ldots \cdot (n-1) \cdot (n-2) \cdot 1 = n!$.

1By definition: $0! = 1$.

One-to-One Functions.

How many one-to-one functions from $|S|$ to $|S|$.

$|S|$ choices for $f(s_1)$, $|S|-1$ choices for $f(s_2)$, ...

So total number is $|S| \times |S|-1 \cdots 1 = |S|$!

A one-to-one function is a permutation!

Functions, polynomials.

How many functions $f$ mapping $S$ to $T$?

$|T|$ ways to choose for $f(s_1)$, $|T|$ ways to choose for $f(s_2)$, ...

$\ldots |T|^{(k)}$.

How many polynomials of degree $d$ modulo $p$?
p ways to choose for first coefficient, $p$ ways for second, ...

$\ldots p^d$.

$p$ values for first point, $p$ values for second, ...

$p^{d+1}$.

Counting sets..when order doesn’t matter.

How many poker hands?

$52 \times 51 \times 50 \times 49 \times 48$ ???

Are A, K, Q, 10, J of spades and ... ways to choose 5 out of 52 possibilities.

When each unordered object corresponds equal numbers of ordered objects.
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.

Stars and Bars.

Simple Practice.

Sampling...

Sample \( k \) items out of \( n \)

Without replacement:

- Order matters: \( n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!} \)
- Order does not matter: \( \binom{n}{k} \) and pronounced “n choose k.”

Stars and Bars.

How many ways to add up \( n \) numbers to equal \( k \)?

Or: \( k \) choices from set of \( n \) possibilities with replacement.

Sample with replacement where order just doesn’t matter.

How many ways can Alice, Bob, and Eve split 5 dollars.

How many red nodes (ordered objects)? 9.
How many red nodes mapped to one blue node? 3.
How many blue nodes (unordered objects)? \( \frac{9}{3} = 3 \).
How many poker deals? \( 52 \times 51 \times 50 \times 49 \times 48 \)
How many poker hands per deal? \( \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} \)
How many poker hands? \( \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} \)

Stars and bars....

How many poker deals?

Total orderings? 5!

First rule:

- How many poker hands? \( 52 \times 51 \times 50 \times 49 \times 48 \)
- Alice: 2, Bob: 1, Eve: 2.

How many poker hands per deal?

Total orderings? How many poker deals?

Second rule:

- How many poker hands per deal?
- Second Rule: divide by number of orderings.

How many poker hands?

Stars and Bars.

Ordered, unordered.

How many poker hands per deal?

Stars and Bars.

Simple Practice.

How many poker hands per deal?

Ordered, except for A!

How many poker hands per deal?

Second Rule:

- How many poker hands per deal? Map each deal to ordered deal.
- \( 5! \)
- \( 52 \times 51 \times 50 \times 49 \times 48 \)
- How many poker hands?
- \( \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} \)

Order matters:

- \( 5! \times 5! \times 2! \)

Order doesn’t matter:

- \( \binom{11}{3} \) and pronounced “n choose k.”

Sampling...

How many ways can Bob and Alice split 5 dollars?

Choose \( k \) out of \( n \)?

Choose 2 out of \( n \)?

Choose 3 out of \( n \)?

Choose \( k \) out of \( n \)?

Notation: \( \binom{n}{k} \) and pronounced “n choose k.”

Ordered to unordered.

How many ways to add up \( n \) numbers to equal \( k \)?

Or: \( k \) choices from set of \( n \) possibilities with replacement.

Sample with replacement where order just doesn’t matter.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!
Stars and Bars.

How many different 5 star and 2 bar diagrams?
7 positions in which to place the 2 bars.
\( \left( \begin{array}{c} 7 \\ 2 \end{array} \right) \) ways to do so and \( \left( \begin{array}{c} 5 \\ 2 \end{array} \right) \) ways to split 5$ among 3 people.
Ways to add up \( n \) numbers to sum to \( k \) or
"\( k \) from \( n \) with replacement where order doesn't matter."
In general, \( k \) stars \( n-1 \) bars.

\[
\star \star | \star | \cdots | \star.
\]
\( n + k - 1 \) positions from which to choose \( n - 1 \) bar positions.

\[
\binom{n+k-1}{n-1}
\]

Summary.

First rule: \( n_1 \times n_2 \times \cdots \times n_k \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sample without replacement: \( \frac{n^k}{\prod_{i=1}^{k} n_i} \).

Second rule: when order doesn’t matter divide..when possible.

Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n^k}{\prod_{i=1}^{k} n_i} \).

"\( n \ choose k \)"

One-to-one rule: equal in number if one-to-one correspondence.
Sample with replacement and order doesn’t matter: \( \binom{k+n-1}{n} \).

See you on Friday.