Today: Proofs!!!

1. By Example.
2. Direct. (Prove $P \implies Q$.)
3. by Contraposition (Prove $P \implies Q$)
4. by Contradiction (Prove $P$.)
5. by Cases
Quick Background and Notation.

Integers closed under addition.
Quick Background and Notation.

Integers closed under addition.

\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]

- Yes! \( 2 \mid 4 \)
- No! \( 7 \mid 23 \)
- No! \( 4 \mid 2 \)

Formally:

\[ a \mid b \iff \exists q \in \mathbb{Z} \text{ where } b = aq \]

- Yes! \( 3 \mid 15 \) since for \( q = 5 \), \( 15 = 3 \times 5 \).

A natural number \( p > 1 \) is prime if it is divisible only by 1 and itself.
Quick Background and Notation.

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\[ a, b \in \mathbb{Z} \implies a + b \in \mathbb{Z} \]

\( a \mid b \) means “\( a \) divides \( b \)”. 

\( 2 \mid 4 \)\( \text{Yes!} \)

\( 7 \mid 23 \)\( \text{No!} \)

\( 4 \mid 2 \)\( \text{No!} \)

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\( 2 \mid 4 ? \)
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\[ a, b \in Z \implies a + b \in Z \]

\( a \mid b \) means “a divides b”.

2|4? Yes!
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A natural number \( p > 1 \), is **prime** if it is divisible only by 1 and itself.
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

**Proof:** Assume $a|b$ and $a|c$.
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a \mid b$ and $a \mid c$ then $a \mid b - c$.

**Proof:** Assume $a \mid b$ and $a \mid c$

$b = aq$
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

**Proof:** Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$

$(b - c) = a(q - q')$ is an integer so $a|b - c$.

Works for $\forall a, b, c \in \mathbb{Z}$. 

Direct Proof Form:

**Goal:** $P \Rightarrow Q$

Assume $P$.

... Therefore $Q$. 


Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | b - c$.

**Proof:** Assume $a | b$ and $a | c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a | b$ and $a | c$ then $a | b - c$.

**Proof:** Assume $a | b$ and $a | c$

\begin{align*}
    b &= aq \\
    c &= aq' \text{ where } q, q' \in \mathbb{Z}
\end{align*}

\begin{align*}
    b - c &= aq - aq' \\
    &= a(q - q')
\end{align*}

Therefore $a | (b - c)$.
**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

**Proof:** Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

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**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

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Theorem: For any $a, b, c \in Z$, if $a \mid b$ and $a \mid c$ then $a \mid b - c$.

Proof: Assume $a \mid b$ and $a \mid c$

$b = aq$ and $c = aq'$ where $q, q' \in Z$

$b - c = aq - aq' = a(q - q')$ Done?

$(b - c) = a(q - q')$
**Theorem:** For any \( a, b, c \in \mathbb{Z} \), if \( a \mid b \) and \( a \mid c \) then \( a \mid b - c \).

**Proof:** Assume \( a \mid b \) and \( a \mid c \)

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(b - c) = a(q - q') \quad \text{and} \quad (q - q') \text{ is an integer so}
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\[b = aq\] and \(c = aq'\) where \(q, q' \in \mathbb{Z}\)

\[b - c = aq - aq' = a(q - q')\] Done?

\((b - c) = a(q - q')\) and \((q - q')\) is an integer so \(a \mid (b - c)\)
**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

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Proof: Assume \( a \mid b \) and \( a \mid c \)

\[ b = aq \quad \text{and} \quad c = aq' \quad \text{where} \quad q, q' \in \mathbb{Z} \]

\[ b - c = aq - aq' = a(q - q') \]

Done?

\[ (b - c) = a(q - q') \] and \((q - q')\) is an integer so

\[ a \mid (b - c) \]

Works for \( \forall a, b, c \)?
Theorem: For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

Proof: Assume $a|b$ and $a|c$
- $b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$
- $b - c = aq - aq' = a(q - q')$ Done?

$(b - c) = a(q - q')$ and $(q - q')$ is an integer so
- $a|(b - c)$

Works for $\forall a, b, c$?
- Argument applies to every $a, b, c \in \mathbb{Z}$. 

\[ \square \]
Direct Proof.

**Theorem:** For any $a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|b - c$.

**Proof:** Assume $a|b$ and $a|c$

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**Theorem:** For any \( a, b, c \in \mathbb{Z} \), if \( a|b \) and \( a|c \) then \( a|b - c \).

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Works for \( \forall a, b, c \)?

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Direct Proof Form:

Goal: \( P \implies Q \)
Direct Proof.

**Theorem:** For any \( a, b, c \in \mathbb{Z} \), if \( a \mid b \) and \( a \mid c \) then \( a \mid b - c \).

**Proof:** Assume \( a \mid b \) and \( a \mid c \)

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Argument applies to every \( a, b, c \in \mathbb{Z} \).

**Direct Proof Form:**

**Goal:** \( P \implies Q \)

Assume \( P \).
Theorem: For any \( a, b, c \in \mathbb{Z} \), if \( a|b \) and \( a|c \) then \( a|b - c \).

Proof: Assume \( a|b \) and \( a|c \)

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b = aq \quad \text{and} \quad c = aq' \quad \text{where} \quad q, q' \in \mathbb{Z}
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\((b - c) = a(q - q')\) and \((q - q')\) is an integer so

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Direct Proof Form:

Goal: \( P \implies Q \)

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\text{Assume } P.
\]

\[
\ldots
\]
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**Proof:** Assume $a|b$ and $a|c$

$b = aq$ and $c = aq'$ where $q, q' \in \mathbb{Z}$

$b - c = aq - aq' = a(q - q')$ Done?

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Direct Proof Form:

Goal: $P \implies Q$

Assume $P$.

... Therefore $Q$. 


Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11 | n$.

∀ $n \in D_3$, $(11 | \text{alt. sum of digits of } n) \Rightarrow 11 | n$.

Examples:

$n = 121$  
Alt. Sum: $1 - 2 + 1 = 0$. Divisible by 11. As is 121.

$n = 605$  

Proof:

For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b$

The left hand side is $n$, $k + 9a + b$ is an integer.

$\Rightarrow 11 | n$.

Direct proof of $P \Rightarrow Q$: Assumed $P$: $11 | a - b + c$.

Proved $Q$: $11 | n$. 
Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

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Examples:

$n = 121$

Alt Sum: $1 - 2 + 1 = 0$.

Divis. by 11.

As is $121 = 11 (11)$

Proof:

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Add $99a + 11b$ to both sides.

$100a + 10b + c = 11k + 99a + 11b$

$= 11 (k + 9a + b)$

Left hand side is $n$, $k + 9a + b$ is integer.

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Direct proof of $P \Rightarrow Q$: Assumed $P$: $11 | a - b + c$.

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Another direct proof.

Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, than $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$
Another direct proof.

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Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11 \mid n$.

$$\forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n$$

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$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. 
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Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$
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**Proof:** For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$. 
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Let $D_3$ be the 3 digit natural numbers.

Theorem: For $n \in D_3$, if the alternating sum of digits of $n$ is divisible by 11, then $11|n$.

$$\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$$

Examples:

$n = 121$  Alt Sum: $1 - 2 + 1 = 0$. Divis. by 11. As is 121.

$n = 605$  Alt Sum: $6 - 0 + 5 = 11$ Divis. by 11. As is $605 = 11(55)$

Proof: For $n \in D_3$, $n = 100a + 10b + c$, for some $a, b, c$.

Assume: Alt. sum: $a - b + c = 11k$ for some integer $k$.

Add $99a + 11b$ to both sides.

$$100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)$$

Left hand side is $n$,
Another direct proof.

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\[
100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b)
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Left hand side is \( n \), \( k + 9a + b \) is integer.
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Another direct proof.

Let \( D_3 \) be the 3 digit natural numbers.

Theorem: For \( n \in D_3 \), if the alternating sum of digits of \( n \) is divisible by 11, than \( 11 \mid n \).

\[ \forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n \]

Examples:
\[ n = 121 \quad \text{Alt Sum: } 1 - 2 + 1 = 0. \text{ Divis. by 11. As is 121.} \]
\[ n = 605 \quad \text{Alt Sum: } 6 - 0 + 5 = 11 \text{ Divis. by 11. As is } 605 = 11(55) \]

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Add \( 99a + 11b \) to both sides.

\[ 100a + 10b + c = 11k + 99a + 11b = 11(k + 9a + b) \]

Left hand side is \( n, k + 9a + b \) is integer. \( \implies 11 \mid n. \)

\[ \square \]
Another direct proof.

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Direct proof of $P \implies Q$: Assumed $P$: $11|a - b + c$.
Another direct proof.

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$\square$ Direct proof of $P \implies Q$: Assumed $P$: $11|a - b + c$. Proved $Q$: $11|n$. 
The Converse

Thm: $\forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n$
The Converse

Thm: \( \forall n \in D_3, (11 \mid \text{alt. sum of digits of } n) \implies 11 \mid n \)

Is converse a theorem?
\( \forall n \in D_3, (11 \mid n) \implies (11 \mid \text{alt. sum of digits of } n) \)
The Converse

Thm: $\forall n \in D_3, (11|\text{alt. sum of digits of } n) \implies 11|n$

Is converse a theorem?
$\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Yes?
The Converse

Thm: $\forall n \in D_3, (11 | \text{alt. sum of digits of } n) \implies 11 | n$

Is converse a theorem?
$\forall n \in D_3, (11 | n) \implies (11 | \text{alt. sum of digits of } n)$

Yes? No?
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 | n) \iff (11 | \text{alt. sum of digits of } n) \)

Proof:

| Theorem: | \( \forall n \in D_3, (11 | n) \iff (11 | \text{alt. sum of digits of } n) \) |
|----------|---------------------------------------------------------------------|
| Proof:   | \[
|         | \text{Assume } 11 | n. } \text{ Let } n = 100a + 10b + c = 11k \Rightarrow 99a + 11b + (a - b + c) = 11k \Rightarrow a - b + c = 11(\text{a number } \ell) \in \mathbb{Z} \\
|         | \text{That is } 11 | \text{alternating sum of digits. } |
|         | \text{Note: similar proof to other. In this case every } \iff \text{ is } \iff. } \text{ Often works with arithmetic properties except when multiplying by 0. } \text{We have. } |
|         | \] |
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 | n) \iff (11 | \text{alt. sum of digits of } n) \)

Proof: Assume 11 | n.
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n) \)
Proof: Assume 11|\(n\).
\[
\begin{align*}
n &= 100a + 10b + c = 11k
\end{align*}
\]
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11|n \).

\[
n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k
\]
Another Direct Proof.

**Theorem:** \( \forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11|n \).

\[
n = 100a + 10b + c = 11k \iff \\
99a + 11b + (a - b + c) = 11k \iff \\
a - b + c = 11k - 99a - 11b
\]
Another Direct Proof.

**Theorem:** \( \forall n \in D_3, (11 \mid n) \implies (11 \mid \text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11 \mid n \).

\[
n = 100a + 10b + c = 11k \implies \\
99a + 11b + (a - b + c) = 11k \implies \\
a - b + c = 11k - 99a - 11b \implies \\
a - b + c = 11(k - 9a - b)
\]
Another Direct Proof.

Theorem: $\forall n \in D_3, (11|n) \iff (11|\text{alt. sum of digits of } n)$

Proof: Assume $11|n$.

\[ n = 100a + 10b + c = 11k \implies \]
\[ 99a + 11b + (a - b + c) = 11k \implies \]
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\[ a - b + c = 11\ell \]
Another Direct Proof.

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\[
\begin{align*}
n &= 100a + 10b + c = 11k \\ 99a + 11b + (a - b + c) &= 11k \\ a - b + c &= 11k - 99a - 11b \\ a - b + c &= 11(k - 9a - b) \\ a - b + c &= 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\end{align*}
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a - b + c &= 11(k - 9a - b) \\
a - b + c &= 11\ell \quad \text{where } \ell = (k - 9a - b) \in \mathbb{Z}
\end{align*}
\]

That is \( 11|\text{alternating sum of digits} \). \( \square \)
Another Direct Proof.

Theorem: ∀n ∈ D₃, (11|n) ⟷ (11|alt. sum of digits of n)

Proof: Assume 11|n.

\[ n = 100a + 10b + c = 11k \Rightarrow \]
\[ 99a + 11b + (a - b + c) = 11k \Rightarrow \]
\[ a - b + c = 11k - 99a - 11b \Rightarrow \]
\[ a - b + c = 11(k - 9a - b) \Rightarrow \]
\[ a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) ∈ Z \]

That is 11|alternating sum of digits.

Note: similar proof to other. In this case every ⟷ is ⇔
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\]

That is 11|alternating sum of digits. \(\square\)

Note: similar proof to other. In this case every \(\implies\) is \(\iff\)

Often works with arithmetic properties except when multiplying by 0.
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11 \mid n) \iff (11 \mid \text{alt. sum of digits of } n) \)

**Proof:** Assume \( 11 \mid n \).

\[
n = 100a + 10b + c = 11k \\
99a + 11b + (a - b + c) = 11k \\
a - b + c = 11k - 99a - 11b \\
a - b + c = 11(k - 9a - b) \\
a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\]

That is \( 11 \mid \text{alternating sum of digits.} \)

\[\square\]

Note: similar proof to other. In this case every \( \iff \) is \( \iff \)

Often works with arithmetic properties except when multiplying by 0.

We have.
Another Direct Proof.

Theorem: \( \forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n) \)

Proof: Assume \(11|n\).

\[
n = 100a + 10b + c = 11k \implies 99a + 11b + (a - b + c) = 11k \implies a - b + c = 11k - 99a - 11b \implies a - b + c = 11(k - 9a - b) \implies a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in \mathbb{Z}
\]

That is \(11|\text{alternating sum of digits}\). \(\square\)

Note: similar proof to other. In this case every \(\implies\) is \(\iff\)

Often works with arithmetic properties except when multiplying by 0.

We have.

Theorem: \( \forall n \in N', (11|\text{alt. sum of digits of } n) \iff (11|n) \)
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \]

What do we know about \( d \)?

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \) ... and prove \( \neg P \).

Conclusion: \( \neg Q = \implies \neg P \) equivalent to \( P = \implies Q \).

Proof:
Assume \( \neg Q \): \( d \) is even.

\[ d = 2k \]

\( d \mid n \) so we have

\[ n = qd = q(2k) = 2(kq) \]

\( n \) is even.

\( \neg P \)
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.
Proof by Contraposition

Thm: For $n \in Z^+$ and $d|n$. If $n$ is odd then $d$ is odd.

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Proof by Contraposition

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What to do?
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

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Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

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Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

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Goal: Prove $P \implies Q$.

Assume $\neg Q$
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What to do?

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Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$
Proof by Contraposition

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$n = 2k + 1$ what do we know about $d$?

What to do?

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...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$. 
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d|n \). If \( n \) is odd then \( d \) is odd.

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What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

Conclusion: \( \neg Q \implies \neg P \) equivalent to \( P \implies Q \).

**Proof:** Assume \( \neg Q \): \( d \) is even.
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

Conclusion: \( \neg Q \implies \neg P \) equivalent to \( P \implies Q \).

Proof: Assume \( \neg Q \): \( d \) is even. \( d = 2k \).
Proof by Contraposition

Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

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\( d \mid n \) so we have
Thm: For \( n \in \mathbb{Z}^+ \) and \( d|n \). If \( n \) is odd then \( d \) is odd.

\[ n = 2k + 1 \] what do we know about \( d \)?

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

Conclusion: \( \neg Q \implies \neg P \) equivalent to \( P \implies Q \).

**Proof:** Assume \( \neg Q \): \( d \) is even. \( d = 2k \).

\( d|n \) so we have

\[ n = qd \]
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d|n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

Goal: Prove $P \implies Q$.

Assume $\neg Q$

...and prove $\neg P$.

Conclusion: $\neg Q \implies \neg P$ equivalent to $P \implies Q$.

Proof: Assume $\neg Q$: $d$ is even. $d = 2k$.

d|n so we have

$n = qd = q(2k)$
Thm: For \( n \in \mathbb{Z}^+ \) and \( d \mid n \). If \( n \) is odd then \( d \) is odd.

\[
n = 2k + 1 \text{ what do we know about } d?\]

What to do?

Goal: Prove \( P \implies Q \).

Assume \( \neg Q \)

...and prove \( \neg P \).

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**Proof:** Assume \( \neg Q \): \( d \) is even. \( d = 2k \).

\( d \mid n \) so we have

\[
n = qd = q(2k) = 2(kq)\]
Proof by Contraposition

Thm: For $n \in \mathbb{Z}^+$ and $d | n$. If $n$ is odd then $d$ is odd.

$n = 2k + 1$ what do we know about $d$?

What to do?

Goal: Prove $P \implies Q$.
Assume $\neg Q$
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\( d|n \) so we have

\[
n = qd = q(2k) = 2(kq) \]

\( n \) is even. \( \neg P \)
Another Contraposition...

Lemma:

For every $n \in \mathbb{N}$, $n^2$ is even $\Rightarrow n$ is even. ($P \Rightarrow Q$)

Proof by contraposition: ($P \Rightarrow Q$) $\equiv$ ($\neg Q \Rightarrow \neg P$)

$P = \text{'}n^2$ is even.' $\ldots$

$\neg P = \text{'}n^2$ is odd' $\ldots$

$Q = \text{'}n$ is even' $\ldots$

$\neg Q = \text{'}n$ is odd'

Prove $\neg Q = \Rightarrow \neg P$:

$n$ is odd $\Rightarrow n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$

$n^2 = 2l + 1$ where $l$ is a natural number.

... and $n^2$ is odd!

$\neg Q = \Rightarrow \neg P$ so $P \Rightarrow Q$ and ...
Lemma: For every $n$ in $N$, $n^2$ is even $\iff n$ is even. ($P \iff Q$)
Lemma: For every $n$ in $N$, $n^2$ is even $\iff n$ is even. ($P \iff Q$)

$n^2$ is even, $n^2 = 2k$, ...
Another Contraposition...

**Lemma:** For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

$n^2$ is even, $n^2 = 2k$, ... $\sqrt{2k}$ even?
Another Contraposition...

**Lemma:** For every \( n \) in \( N \), \( n^2 \) is even \( \iff \) \( n \) is even. \((P \iff Q)\)

**Proof by contraposition:** \((P \iff Q) \equiv (\neg Q \iff \neg P)\)
Lemma: For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \iff Q$)

Proof by contraposition: ($P \iff Q$) $\equiv$ ($\neg Q \iff \neg P$)

$P = 'n^2$ is even.' ............
Another Contrapostion...

**Lemma:** For every \( n \) in \( N \), \( n^2 \) is even \( \implies n \) is even. \((P \implies Q)\)

**Proof by contraposition:** \((P \implies Q) \equiv (\neg Q \implies \neg P)\)

\( P = \text{'}n^2 \text{ is even.'} \) ....... \( \neg P = \text{'}n^2 \text{ is odd'} \)
Another Contrapostion...

**Lemma:** For every \( n \) in \( N \), \( n^2 \) is even \( \implies n \) is even. \((P \implies Q)\)

**Proof by contraposition:** \((P \implies Q) \equiv (\neg Q \implies \neg P)\)

\( P = 'n^2 \) is even.' ............ \( \neg P = 'n^2 \) is odd'

\( Q = 'n \) is even' ............
Another Contrapostion...

**Lemma:** For every $n$ in $\mathbb{N}$, $n^2$ is even $\implies$ $n$ is even. ($P \implies Q$)

**Proof by contraposition:** ($P \implies Q$) $\equiv$ ($\neg Q \implies \neg P$)

$P = 'n^2$ is even.’ ............ $\neg P = 'n^2$ is odd’

$Q = 'n$ is even’ ............ $\neg Q = 'n$ is odd’
Lemma: For every $n$ in $N$, $n^2$ is even $\implies n$ is even. ($P \implies Q$)

Proof by contraposition: ($P \implies Q) \equiv (\neg Q \implies \neg P$)

$P = 'n^2$ is even.' ............ $\neg P = 'n^2$ is odd'$

$Q = 'n$ is even' ............ $\neg Q = 'n$ is odd'$

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd.
Lemma: For every \( n \) in \( N \), \( n^2 \) is even \( \implies \) \( n \) is even. \((P \implies Q)\)

Proof by contraposition: \((P \implies Q) \equiv (\neg Q \implies \neg P)\)

\( P = 'n^2 \) is even.' ............ \( \neg P = 'n^2 \) is odd'

\( Q = 'n \) is even' ............ \( \neg Q = 'n \) is odd'

Prove \( \neg Q \implies \neg P: n \) is odd \( \implies \) \( n^2 \) is odd.

\( n = 2k + 1 \)
Lemma: For every \( n \) in \( N \), \( n^2 \) is even \( \implies \) \( n \) is even. \((P \implies Q)\)

Proof by contraposition: \((P \implies Q) \equiv (\neg Q \implies \neg P)\)
\( P = 'n^2 \text{ is even}' \) .............. \( \neg P = 'n^2 \text{ is odd}' \)
\( Q = 'n \text{ is even}' \) .............. \( \neg Q = 'n \text{ is odd}' \)
Prove \( \neg Q \implies \neg P: n \text{ is odd } \implies n^2 \text{ is odd}. \)
\( n = 2k + 1 \)
\( n^2 = 4k^2 + 4k + 1 = 2(2k+k)+1. \)
Lemma: For every $n$ in $N$, $n^2$ is even $\iff$ $n$ is even. ($P \iff Q$)

Proof by contraposition: ($P \iff Q) \equiv (\neg Q \iff \neg P)$

$P = 'n^2$ is even.’ ............ $\neg P = 'n^2$ is odd’

$Q = 'n$ is even’ ............ $\neg Q = 'n$ is odd’

Prove $\neg Q \iff \neg P$: $n$ is odd $\iff$ $n^2$ is odd.

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1.$

$n^2 = 2l + 1$ where $l$ is a natural number..
Lemma: For every \( n \) in \( N \), \( n^2 \) is even \( \iff \) \( n \) is even. \((P \iff Q)\)

Proof by contraposition: \((P \iff Q) \equiv (\neg Q \iff \neg P)\)

\( P = 'n^2 \) is even.' ............. \( \neg P = 'n^2 \) is odd'

\( Q = 'n \) is even' ............. \( \neg Q = 'n \) is odd'

Prove \( \neg Q \iff \neg P: n \) is odd \( \iff \) \( n^2 \) is odd.

\[ n = 2k + 1 \]

\[ n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1. \]

\[ n^2 = 2l + 1 \] where \( l \) is a natural number..

... and \( n^2 \) is odd!
Lemma: For every \( n \) in \( N \), \( n^2 \) is even \( \implies \) \( n \) is even. \((P \implies Q)\)

Proof by contraposition: \((P \implies Q) \equiv (\neg Q \implies \neg P)\)

\( P = \text{'}n^2 \text{ is even.'} \) ............ \( \neg P = \text{'}n^2 \text{ is odd'} \)

\( Q = \text{'}n \text{ is even'} \) ............ \( \neg Q = \text{'}n \text{ is odd'} \)

Prove \( \neg Q \implies \neg P: \) \( n \) is odd \( \implies \) \( n^2 \) is odd.

\( n = 2k + 1 \)

\( n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1. \)

\( n^2 = 2l + 1 \) where \( l \) is a natural number..

... and \( n^2 \) is odd!

\( \neg Q \implies \neg P \)
Another Contraposition...

**Lemma:** For every \( n \) in \( N \), \( n^2 \) is even \( \implies \) \( n \) is even. (\( P \implies Q \))

**Proof by contraposition:** (\( P \implies Q \)) \( \equiv \) (\( \neg Q \implies \neg P \))

\( P = 'n^2 \) is even.’ ........... \( \neg P = 'n^2 \) is odd’

\( Q = 'n \) is even’ ........... \( \neg Q = 'n \) is odd’

Prove \( \neg Q \implies \neg P \): \( n \) is odd \( \implies \) \( n^2 \) is odd.

\( n = 2k + 1 \)

\( n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1. \)

\( n^2 = 2l + 1 \) where \( l \) is a natural number..

... and \( n^2 \) is odd!

\( \neg Q \implies \neg P \) so \( P \implies Q \) and ...
Lemma: For every $n$ in $N$, $n^2$ is even $\implies n$ is even. $(P \implies Q)$

Proof by contraposition: $(P \implies Q) \equiv (\neg Q \implies \neg P)$

$P = 'n^2$ is even.' ........... $\neg P = 'n^2$ is odd' 

$Q = 'n$ is even' ............ $\neg Q = 'n$ is odd' 

Prove $\neg Q \implies \neg P$: $n$ is odd $\implies n^2$ is odd. 

$n = 2k + 1$

$n^2 = 4k^2 + 4k + 1 = 2(2k + k) + 1$.

$n^2 = 2l + 1$ where $l$ is a natural number.

... and $n^2$ is odd!

$\neg Q \implies \neg P$ so $P \implies Q$ and ...
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show:
Proof by contradiction: form

**Theorem:** \(\sqrt{2}\) is irrational.

Must show: For every \(a, b \in \mathbb{Z}\),
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in Z \), \( (\frac{a}{b})^2 \neq 2 \).
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$. 
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \ldots$
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** \( P \).

\( \neg P \implies P_1 \cdots \implies R \)
Proof by contradiction:form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $\left(\frac{a}{b}\right)^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P$
Proof by contradiction: form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies P_1$
**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in Z$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies P_1 \cdots$
**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** \( P \).

\[ \neg P \implies P_1 \cdots \implies R \]

\[ \neg P \implies P_1 \cdots \implies \neg R \]
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( (\frac{a}{b})^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** \( P \).

\[ \neg P \implies P_1 \cdots \implies R \]

\[ \neg P \implies P_1 \cdots \implies \neg R \]

\[ \neg P \implies \text{False} \]
Proof by contradiction:form

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies P_1 \cdots \implies \neg R$

$\neg P \implies \text{False}$

Contrapositive: True $\implies P$. 

Proof by contradiction:

**Theorem:** $\sqrt{2}$ is irrational.

Must show: For every $a, b \in \mathbb{Z}$, $(\frac{a}{b})^2 \neq 2$.

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** $P$.

$\neg P \implies P_1 \cdots \implies R$

$\neg P \implies P_1 \cdots \implies \neg R$

$\neg P \implies \text{False}$

Contrapositive: True $\implies P$. Theorem $P$ is proven.
Proof by contradiction: form

**Theorem:** \( \sqrt{2} \) is irrational.

Must show: For every \( a, b \in \mathbb{Z} \), \( \left( \frac{a}{b} \right)^2 \neq 2 \).

A simple property (equality) should always “not” hold.

Proof by contradiction:

**Theorem:** \( P \).

\[ \neg P \implies P_1 \cdots \implies R \]

\[ \neg P \implies P_1 \cdots \implies \neg R \]

\[ \neg P \implies \text{False} \]

Contrapositive: \( \text{True} \implies P \). Theorem \( P \) is proven.
Theorem: \( \sqrt{2} \) is irrational.
Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$:

Reduced form: $a$ and $b$ have no common factors.

$\sqrt{2}b = \frac{a^2}{b^2} = 4k^2$

$a^2$ is even $\Rightarrow a$ is even.

$a = 2k$ for some integer $k$

$b^2 = 2k^2$

$b^2$ is even $\Rightarrow b$ is even.

$a$ and $b$ have a common factor. Contradiction.
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$. 

Contradiction
Theorem: \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.
**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$
Contradiction

Theorem: \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.

\[
\sqrt{2}b = a
\]

\[
2b^2 = a^2
\]
Contradiction

Theorem: $\sqrt{2}$ is irrational.
Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.
Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2$$

$a^2$ is even $\implies$ $a$ is even.
Theorem: \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.

\[
\sqrt{2}b = a
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2b^2 = a^2
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\( a^2 \) is even \( \implies a \) is even.

\( a = 2k \) for some integer \( k \)
Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2 = 4k^2$$

$a^2$ is even $\iff$ $a$ is even.

$a = 2k$ for some integer $k$
Contradiction

**Theorem:** $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

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\sqrt{2}b = a
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2b^2 = a^2 = 4k^2
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$a^2$ is even $\implies$ $a$ is even.

$a = 2k$ for some integer $k$

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b^2 = 2k^2
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Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

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\sqrt{2}b = a
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2b^2 = a^2 = 4k^2
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$a^2$ is even $\iff$ $a$ is even.

$a = 2k$ for some integer $k$

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b^2 = 2k^2
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$b^2$ is even $\iff$ $b$ is even.
Contradiction

**Theorem:** \( \sqrt{2} \) is irrational.

Assume \( \neg P: \sqrt{2} = a/b \) for \( a, b \in \mathbb{Z} \).

Reduced form: \( a \) and \( b \) have no common factors.

\[
\sqrt{2}b = a
\]

\[
2b^2 = a^2 = 4k^2
\]

\( a^2 \) is even \( \implies \) \( a \) is even.

\( a = 2k \) for some integer \( k \)

\[
b^2 = 2k^2
\]

\( b^2 \) is even \( \implies \) \( b \) is even.

\( a \) and \( b \) have a common factor. Contradiction.
Theorem: $\sqrt{2}$ is irrational.

Assume $\neg P$: $\sqrt{2} = a/b$ for $a, b \in \mathbb{Z}$.

Reduced form: $a$ and $b$ have no common factors.

$$\sqrt{2}b = a$$

$$2b^2 = a^2 = 4k^2$$

$a^2$ is even $\implies$ $a$ is even.

$a = 2k$ for some integer $k$

$$b^2 = 2k^2$$

$b^2$ is even $\implies$ $b$ is even.

$a$ and $b$ have a common factor. Contradiction.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.
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**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  
  \[ q = p_1 \times p_2 \times \cdots p_k + 1. \]
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  
  $$q = p_1 \times p_2 \times \cdots p_k + 1.$$  

- $q$ cannot be one of the primes as it is larger than any $p_i$. 

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider $q = p_1 \times p_2 \times \cdots p_k + 1$.

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  
  $$q = p_1 \times p_2 \times \cdots p_k + 1.$$

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$.

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider

  $$q = p_1 \times p_2 \times \cdots p_k + 1.$$

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = $R$) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$,

The original assumption that "the theorem is false" is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider 
  \[ q = p_1 \times p_2 \times \cdots p_k + 1. \]

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1" = R")$ which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdot \cdots p_k$ and $q$, and divides $x - q$,

\[ \implies p | x - q \]
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider
  \[
  q = p_1 \times p_2 \times \cdots p_k + 1.
  \]

- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) (”\( p > 1 \)” = R) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdots p_k \) and \( q \), and divides \( x - q \),
- \( \implies p | x - q \implies p \leq x - q \)

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = p_1 \times p_2 \times \cdots p_k + 1 \).

- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \" = \text{R} \) ) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdot \cdots p_k \) and \( q \), and divides \( x - q \),
- \( \implies p | x - q \implies p \leq x - q = 1. \)
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = p_1 \times p_2 \times \cdots p_k + 1 \).

- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \ ("p > 1" = \text{R}) \) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdots p_k \) and \( q \), and divides \( x - q \), \( \implies p | x - q \implies p \leq x - q = 1 \).
- so \( p \leq 1 \).
Theorem: There are infinitely many primes.

Proof:

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = p_1 \times p_2 \times \cdots p_k + 1 \).

- \( q \) cannot be one of the primes as it is larger than any \( p_i \).
- \( q \) has prime divisor \( p \) ("\( p > 1 \) = R\) which is one of \( p_i \).
- \( p \) divides both \( x = p_1 \cdot p_2 \cdots p_k \) and \( q \), and divides \( x - q \),
- \( \implies p|x - q \implies p \leq x - q = 1 \).
- so \( p \leq 1 \). (Contradicts \( R \).)
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: $p_1, \ldots, p_k$.
- Consider
  
  \[ q = p_1 \times p_2 \times \cdots p_k + 1. \]

- $q$ cannot be one of the primes as it is larger than any $p_i$.
- $q$ has prime divisor $p$ ("$p > 1$" = R) which is one of $p_i$.
- $p$ divides both $x = p_1 \cdot p_2 \cdots p_k$ and $q$, and divides $x - q$,
- $\implies p | x - q \implies p \leq x - q = 1$.
- so $p \leq 1$. (**Contradicts** R.)

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Proof by contradiction: example

**Theorem:** There are infinitely many primes.

**Proof:**

- Assume finitely many primes: \( p_1, \ldots, p_k \).
- Consider \( q = p_1 \times p_2 \times \cdots \times p_k + 1 \).

  - \( q \) cannot be one of the primes as it is larger than any \( p_i \).
  - \( q \) has prime divisor \( p \) (”\( p > 1 \)” = R ) which is one of \( p_i \).
  - \( p \) divides both \( x = p_1 \cdot p_2 \cdot \cdots \cdot p_k \) and \( q \), and divides \( x - q \),
  - \( p \mid x - q \implies p \leq x - q = 1 \).
  - so \( p \leq 1 \). (**Contradicts R.**)

The original assumption that “the theorem is false” is false, thus the theorem is proven.
Product of first $k$ primes..

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
Did we prove?

- “The product of the first \( k \) primes plus 1 is prime.”
- No.
Product of first $k$ primes.

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.
Product of first $k$ primes.

Did we prove?

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Consider example..
Product of first $k$ primes..

Did we prove?

- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..

- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
Did we prove?

- “The product of the first \( k \) primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..

- \( 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509 \)
- There is a prime \textit{in between} \( 13 \) and \( q = 200031 \) that divides \( q \).
Did we prove?
- “The product of the first $k$ primes plus 1 is prime.”
- No.
- The chain of reasoning started with a false statement.

Consider example..
- $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$
- There is a prime *in between* 13 and $q = 200031$ that divides $q$.
- Proof assumed no primes *in between*. 
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.
Proof by cases.

Theorem: \( x^5 - x + 1 = 0 \) has no solution in the rationals.

Proof: First a lemma...
Lemma: If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even!
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma \implies no rational solution.
Proof by cases.

Theorem: $x^5 - x + 1 = 0$ has no solution in the rationals.

Proof: First a lemma...
Lemma: If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma $\implies$ no rational solution.

Proof of lemma: Assume a solution of the form $a/b$. 

Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( a/b \).

\[
\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
\]
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma $\implies$ no rational solution.

**Proof of lemma:** Assume a solution of the form $a/b$.

$$(\frac{a}{b})^5 - \frac{a}{b} + 1 = 0$$

multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

Reduced form $\frac{a}{b}$: $a$ and $b$ can’t both be even! + Lemma $\implies$ no rational solution.

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multiply by $b^5$,

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: $a$ odd, $b$ odd: odd - odd + odd = even.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

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multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]

**Case 1:** \( a \) odd, \( b \) odd: odd - odd + odd = even. Not possible.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

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\]

Multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: \( a \) odd, \( b \) odd: odd - odd + odd = even. Not possible.
Case 2: \( a \) even, \( b \) odd: even - even + odd = even.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

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Multiply by $b^5$,

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a^5 - ab^4 + b^5 = 0
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**Case 1:** $a$ odd, $b$ odd: odd - odd + odd = even. Not possible.

**Case 2:** $a$ even, $b$ odd: even - even + odd = even. Not possible.
Proof by cases.

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**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

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multiply by $b^5$,

$$a^5 - ab^4 + b^5 = 0$$

Case 1: $a$ odd, $b$ odd: odd - odd + odd = even. Not possible.
Case 2: $a$ even, $b$ odd: even - even + odd = even. Not possible.
Case 3: $a$ odd, $b$ even: odd - even + even = even.
Proof by cases.

**Theorem:** $x^5 - x + 1 = 0$ has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

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multiply by $b^5$,

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Proof by cases.

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\left( \frac{a}{b} \right)^5 - a/b + 1 = 0
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multiply by \( b^5 \),

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Case 3: \( a \) odd, \( b \) even: odd - even +even = even. Not possible.
Case 4: \( a \) even, \( b \) even: even - even +even = even.
Proof by cases.

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**Lemma:** If $x$ is a solution to $x^5 - x + 1 = 0$ and $x = a/b$ for $a, b \in \mathbb{Z}$, then both $a$ and $b$ are even.

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\left(\frac{a}{b}\right)^5 - \frac{a}{b} + 1 = 0
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multiply by $b^5$,

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a^5 - ab^4 + b^5 = 0
\]

**Case 1:** $a$ odd, $b$ odd: odd - odd +odd = even. Not possible.
**Case 2:** $a$ even, $b$ odd: even - even +odd = even. Not possible.
**Case 3:** $a$ odd, $b$ even: odd - even +even = even. Not possible.
**Case 4:** $a$ even, $b$ even: even - even +even = even. Possible.
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

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Case 2: \( a \) even, \( b \) odd: even - even +odd = even. Not possible.
Case 3: \( a \) odd, \( b \) even: odd - even +even = even. Not possible.
Case 4: \( a \) even, \( b \) even: even - even +even = even. Possible.

The fourth case is the only one possible,
Proof by cases.

**Theorem:** \( x^5 - x + 1 = 0 \) has no solution in the rationals.

**Proof:** First a lemma...

**Lemma:** If \( x \) is a solution to \( x^5 - x + 1 = 0 \) and \( x = a/b \) for \( a, b \in \mathbb{Z} \), then both \( a \) and \( b \) are even.

Reduced form \( \frac{a}{b} \): \( a \) and \( b \) can’t both be even! + Lemma \( \implies \) no rational solution.

**Proof of lemma:** Assume a solution of the form \( \frac{a}{b} \).

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\left( \frac{a}{b} \right)^5 - \frac{a}{b} + 1 = 0
\]

multiply by \( b^5 \),

\[
a^5 - ab^4 + b^5 = 0
\]

Case 1: \( a \) odd, \( b \) odd: odd - odd + odd = even. Not possible.

Case 2: \( a \) even, \( b \) odd: even - even + odd = even. Not possible.

Case 3: \( a \) odd, \( b \) even: odd - even + even = even. Not possible.

Case 4: \( a \) even, \( b \) even: even - even + even = even. Possible.

The fourth case is the only one possible, so the lemma follows.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

**Case 1:** $x^y = \sqrt{2} \cdot \sqrt{2}$ is rational.

Done!

**Case 2:** $\sqrt{2} \cdot \sqrt{2}$ is irrational.

▶ New values: $x = \sqrt{2} \cdot \sqrt{2}$, $y = \sqrt{2}$.

$\sqrt{2} \cdot \sqrt{2} = (\sqrt{2} \cdot \sqrt{2}) \cdot \sqrt{2} = \sqrt{2}^2 = 2$.

Thus, in this case, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

One of the cases is true so theorem holds.

**Question:** Which case holds?

Don't know!!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational. Let $x = y = \sqrt{2}$. 
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^\sqrt{2}$ is rational. Done!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2^2}$ is rational. Done!

Case 2: $\sqrt{2}^\sqrt{2}$ is irrational.

- New values: $x = \sqrt{2}^\sqrt{2}$, $y = \sqrt{2}$.

$$x^y = \left(\sqrt{2}^\sqrt{2}\right)^\sqrt{2}$$
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2^\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2^{\sqrt{2}}}$ is irrational.

► New values: $x = \sqrt{2^{\sqrt{2}}}$, $y = \sqrt{2}$.

►

$$x^y = \left(\sqrt{2^{\sqrt{2}}}\right)^{\sqrt{2}} = \sqrt{2^{\sqrt{2} \cdot \sqrt{2}}}$$

Thus, in this case, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., $\sqrt{2^2}$).

One of the cases is true so theorem holds.

Question: Which case holds?

Don't know!!
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

\[
x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2.
\]
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$.

Thus, in this case, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^\sqrt{2}$ is rational. Done!

Case 2: $\sqrt{2}^\sqrt{2}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

  $$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^2 = 2.$$

Thus, in this case, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

One of the cases is true so theorem holds.
Proof by cases.

**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

Thus, in this case, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., 2).

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**Theorem:** There exist irrational $x$ and $y$ such that $x^y$ is rational.

Let $x = y = \sqrt{2}$.

Case 1: $x^y = \sqrt{2}^{\sqrt{2}}$ is rational. Done!

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

- New values: $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$.

  \[
x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2.
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Thus, in this case, we have irrational $x$ and $y$ with a rational $x^y$ (i.e., $2$).

One of the cases is true so theorem holds.

Question: Which case holds?
Proof by cases.

**Theorem:** There exist irrational \( x \) and \( y \) such that \( x^y \) is rational.

Let \( x = y = \sqrt{2} \).

Case 1: \( x^y = \sqrt{2}^{\sqrt{2}} \) is rational. Done!

Case 2: \( \sqrt{2}^{\sqrt{2}} \) is irrational.

- New values: \( x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2} \).
- \[
x^y = \left( \sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \times \sqrt{2}} = \sqrt{2}^2 = 2.
\]

Thus, in this case, we have irrational \( x \) and \( y \) with a rational \( x^y \) (i.e., 2).

One of the cases is true so theorem holds.

Question: Which case holds? Don’t know!!!
Be careful.

**Theorem:** $3 = 4$

**Proof:**

Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get $4 = 3$.

By commutativity, the theorem holds.

Don't assume what you want to prove!

**Theorem:** $1 = 2$

**Proof:**

For $x = y$, we have

\[(x^2 - y^2) = x^2 - y^2\]

\[x(x-y) = (x+y)(x-y)\]

\[x = x+y\]

\[x = 2\]

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \Rightarrow Q$ does not mean $Q \Rightarrow P$. 
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. 

Dividing by zero is no good. Also: Multiplying inequalities by a negative.$P \Rightarrow Q$ does not mean $Q \Rightarrow P$.
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. 

- Dividing one side by 3 and the other by 4 to get $4 = 3$.
- By commutativity theorem holds.
- Don't assume what you want to prove!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have $x^2 - y^2 = x(x - y) = (x + y)(x - y)$.

- $x = (x + y)$
- $x = 2$

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \Rightarrow Q$ does not mean $Q \Rightarrow P$. 

Theorem: $3 = 4$

Proof: Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. 

Dividing by zero is no good. Also: Multiplying inequalities by a negative. $P \implies Q$ does not mean $Q \implies P$. 

Be careful.
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity

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**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds.
Theorem: $3 = 4$

Theorem: $3 = 4$


Don’t assume what you want to prove!
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. □

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Be careful.

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Don’t assume what you want to prove!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have
Be careful.

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**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

$$(x^2 - y) = x^2 - y^2$$
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. □

Don’t assume what you want to prove!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

\[
(x^2 - y) = x^2 - y^2
\]

\[
x(x - y) = (x + y)(x - y)
\]
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. □

Don’t assume what you want to prove!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

\[
(x^2 - y) = x^2 - y^2 \\
x(x - y) = (x + y)(x - y) \\
x = (x + y)
\]
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. \[\square\]

Don’t assume what you want to prove!

**Theorem:** $1 = 2$

**Proof:** For $x = y$, we have

\[
\begin{align*}
(x^2 - y) &= x^2 - y^2 \\
x(x - y) &= (x + y)(x - y) \\
x &= (x + y) \\
x &= 2x
\end{align*}
\]

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Theorem: $3 = 4$


Don’t assume what you want to prove!

Theorem: $1 = 2$

Proof: For $x = y$, we have

\[(x^2 - y) = x^2 - y^2\]
\[x(x - y) = (x + y)(x - y)\]
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\[x = 2x\]

$1 = 2$
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. □

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1 = 2
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Theorem: $3 = 4$

Proof: Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. Don’t assume what you want to prove!

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Dividing by zero is no good.
Be careful.

**Theorem:** $3 = 4$

**Proof:** Assume $3 = 4$. Start with $12 = 12$. Divide one side by 3 and the other by 4 to get $4 = 3$. By commutativity theorem holds. □

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1 &= 2
\end{align*}
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Don’t assume what you want to prove!

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**Proof:** For $x = y$, we have

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$$

$$
x(x - y) = (x + y)(x - y)
$$

$$
x = (x + y)
$$

$$
x = 2x
$$

$$
1 = 2
$$

Dividing by zero is no good.

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$. 
Summary

Direct Proof:

To Prove: \( P \Rightarrow Q \).

Assume \( P \).

Prove \( Q \).

By Contraposition:

To Prove: \( P \Rightarrow Q \).

Assume \( \neg Q \).

Prove \( \neg P \).

By Contradiction:

To Prove: \( P \).

Assume \( \neg P \).

Prove False.

By Cases: informal.

Universal: show that statement holds in all cases.
Existence: used cases where one is true.

Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
or \( \sqrt{2} \) and \( \sqrt{2} \) worked.

Careful when proving!

Don't assume the theorem.
Divide by zero.
Watch converse.

... And finally.

Have a nice weekend!!
Summary

Direct Proof:
To Prove: $P \implies Q$.
Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. 

Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$. 

By Contraposition:
To Prove: $P \implies Q$.
Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$.
Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.

Either $\sqrt{2}$ and $\sqrt{2}$ worked.
or $\sqrt{2}$ and $\sqrt{2}$ worked.

Careful when proving!
Don’t assume the theorem.
Divide by zero.
Watch converse.

And finally.
Have a nice weekend!!
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:

... And finally.
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Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$

By Contradiction:
To Prove: $P$
Assume $\neg P$.
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Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$.
Summary

Direct Proof:
   To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
   To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \). Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
Summary

Direct Proof:
   To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
   To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
   To Prove: \( P \)

Either \( \sqrt{2} \) and \( \sqrt{2} \) worked. or \( \sqrt{2} \) and \( \sqrt{2} \) worked.

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Summary

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To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$.

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To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \). Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \). Assume \( \neg P \). Prove \( \text{False} \).
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove \text{False}.

By Cases: informal.

...And finally.

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Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove $\text{False}$.

By Cases: informal.
Universal: show that statement holds in all cases.
Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Summary

Direct Proof:
   To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
   To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
   To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
   Universal: show that statement holds in all cases.
   Existence: used cases where one is true.
   Either $\sqrt{2}$ and $\sqrt{2}$ worked.

Careful when proving!
  Don't assume the theorem.
  Divide by zero.
  Watch converse.

And finally.
Have a nice weekend!!
Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either $\sqrt{2}$ and $\sqrt{2}$ worked.
  or $\sqrt{2}$ and $\sqrt{2}^{\sqrt{2}}$ worked.
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove \text{False} .

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
  Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
  or \( \sqrt{2} \) and \( \sqrt{2} \sqrt{2} \) worked.

Careful when proving!
Don't assume the theorem.
Divide by zero.
Watch converse.

And finally.
Have a nice weekend!!
Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either $\sqrt{2}$ and $\sqrt{2}$ worked.
   or $\sqrt{2}$ and $\sqrt{2}^2$ worked.

Careful when proving!
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
or \( \sqrt{2} \) and \( \sqrt{2} \sqrt{2} \) worked.

Careful when proving!
Don’t assume the theorem.
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False.

By Cases: informal.
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Existence: used cases where one is true.
Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
or \( \sqrt{2} \) and \( \sqrt{2} \sqrt{2} \) worked.

Careful when proving!
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Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either $\sqrt{2}$ and $\sqrt{2}$ worked.
  or $\sqrt{2}$ and $\sqrt{2\sqrt{2}}$ worked.

Careful when proving!
Don’t assume the theorem. Divide by zero. Watch converse.
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
   or \( \sqrt{2} \) and \( \sqrt{2} \) worked.

Careful when proving!
Don’t assume the theorem. Divide by zero. Watch converse. ...
Summary

Direct Proof:
To Prove: \( P \implies Q \). Assume \( P \). Prove \( Q \).

By Contraposition:
To Prove: \( P \implies Q \) Assume \( \neg Q \). Prove \( \neg P \).

By Contradiction:
To Prove: \( P \) Assume \( \neg P \). Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either \( \sqrt{2} \) and \( \sqrt{2} \) worked.
   or \( \sqrt{2} \) and \( \sqrt{2} \sqrt{2} \) worked.

Careful when proving!
   Don’t assume the theorem. Divide by zero. Watch converse. ...

And finally.
Summary

Direct Proof:
To Prove: $P \implies Q$. Assume $P$. Prove $Q$.

By Contraposition:
To Prove: $P \implies Q$ Assume $\neg Q$. Prove $\neg P$.

By Contradiction:
To Prove: $P$ Assume $\neg P$. Prove False.

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
   Either $\sqrt{2}$ and $\sqrt{2}$ worked.
      or $\sqrt{2}$ and $\sqrt{2}\sqrt{2}$ worked.

Careful when proving!
   Don’t assume the theorem. Divide by zero. Watch converse. ...

And finally. Have a nice weekend!!