What do we learn from observations?

1. Probability Basics Review
2. Examples
3. Conditional Probability

Probability Basics Review

Setup:
- Random Experiment.
  Flip a coin twice.
- Probability Space.
  - Sample Space: $\Omega = \{HH, HT, TH, TT\}$.
  - $\Omega = \{HH, HT, TH, TT\}$.
  - Probability: $\Pr[\omega]$ for all $\omega \in \Omega$.
    - $\Pr[HH] = \cdots = \Pr[TT] = \frac{1}{4}$.
    - $\sum_{\omega \in \Omega} \Pr[\omega] = 1$.
- Event: $A \subseteq \Omega$, $\Pr[A] = \sum_{\omega \in \Omega} \Pr[\omega]$.
  - $\Pr[\text{at least one H out of two tosses}] = \Pr[HT, TH, HH] = \frac{3}{4}$

Calculation.

Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$ 

$$\binom{2n}{n} \approx \frac{(2\pi n)^{n/2}}{\sqrt{e}} \approx \frac{4^n}{\sqrt{n!}}.$$ 

$$\Pr[E] \approx \frac{\binom{50}{25}}{2^{100}} = \frac{1}{\sqrt{50\pi}} \approx 0.08.$$ 

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$.

Uniform probability space: $\Pr[\omega] = \frac{1}{2^{100}}$.

Event $E = \{\text{100 coin tosses with exactly 50 heads}\}$?

Choose 50 positions out of 100 to be heads.

$$|E| = \binom{100}{50}.$$ 

$$\Pr[E] = \frac{\binom{50}{50}}{2^{100}} = \frac{1}{2^{100}}$$

Probability of even number of heads in 57 coin tosses

$\Omega = 57$ coin tosses. $|\Omega| = 2^{57}$.

Let $E = \{\omega \in \Omega | \text{number of Hs in } \omega \text{ is even}\}$.

Fact: $P(E) = 1/2$.

Proof:

Consider the correspondence:

$$HA_1 A_2 \cdots A_{56} \leftrightarrow TA_1 A_2 \cdots A_{56}.$$ 

Here, $A_1, \ldots, A_{56}$ are 56 coin flips.

It matches every even sequence to an odd sequence, and conversely.

Hence, there are exactly as many odd as even sequences.

$$|E| = |\Omega|/2 \Rightarrow P(E) = |E| = \frac{1}{2}$$
Probability more heads than tails in 100 coin tosses.

Ω = 100 coin tosses.
|Ω| = 2^{100}.

Recall event E = ‘equal heads and tails’
Event F = ‘more heads than tails’
Event G = ‘more tails than heads’

A 1-to-1 correspondence between outcomes in F and G!
|F| = |Ω|.
E, F and G are disjoint.
Pr[ω] = 1/36 for all ω.

What is the probability of 1. the red die showing 6?
E1 = {(6, b) : 1 ≤ b ≤ 6}, |E1| = 6, Pr[E1] = |E1|/|Ω| = 1/6
2. at least one die showing 6?
E2 = {(a, 6) : 1 ≤ a ≤ 6} = blue die shows 6
E = E1 ∪ E2 = red or blue die (or both) show 6

Pr[E] = |E|/|Ω| = 11/36

Roll a red and a blue die.
Ω = {(a, b) : 1 ≤ a, b ≤ 6} = {1, 2, . . . , 6}^2.
Uniform: Pr[ω] = 1/36 for all ω.

What is the probability of
1. the dice sum to 7?
E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}: |E| = 6.

Counting argument:
for each choice a of the value of the red die,
there is exactly one choice b = 7 − a for the blue die,
so there are 6 total choices.


2. the dice sum to 10?
E = {(4, 6), (5, 5), (6, 4)}

Roll a red and a blue die.

Roll two blue dice.

The key idea is that we do not distinguish the dice.
Roll die 1, then die 2. Then forget the order.

For instance, we consider that \((2, 5)\) and \((5, 2)\) are the same outcome.

Thus, 
\[
\Omega' = \{(a, b) \mid 1 \leq a \leq b \leq 6\}.
\]
We see that 
\[
\Pr[(1, 3)] = \frac{2}{36} \quad \text{and} \quad \Pr[(2, 2)] = \frac{1}{36}.
\]

Now what is the probability of at least one die showing 6?

In \(\Omega\), \(\Pr[A] = \frac{11}{36}\); in \(\Omega'\), \(\Pr[B] = \frac{5}{36} \times \frac{2}{36} + \frac{1}{36} \times \frac{3}{36}\).

Of course, this is the same as for distinguishable dice!
The event does not depend on the dice being distinguishable.

What is the probability of the dice sum to 7?

In \(\Omega\), \(\Pr[A] = \frac{6}{36}\); in \(\Omega'\), \(\Pr[B] = \frac{3}{36}\).

Of course, this is the same as for distinguishable dice!
The event does not depend on the dice being distinguishable.

Really not uniform! and not finite!

▶ Experiment: Toss three times a coin with \(\Pr[H] = \frac{2}{3}\).

- \(\Omega = \{HHH, HHT, HTT, THH, THT, TTH, TTT\}\).
- \(\Pr[HHH] = \left(\frac{2}{3}\right)^3\); \(\Pr[HHT] = \left(\frac{2}{3}\right)^2 \times \frac{1}{3}\); . . .

- Toss a fair coin until you get a heads.

- \(\Omega = \{H, TH, TTH, TTT\ldots\}

- \(\Pr[H] = \frac{1}{2}\); \(\Pr[TH] = \frac{1}{2}\times \Pr[TTH] = \frac{1}{8}\)

- Still sums to 1. Indeed \(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1\).
Set notation review

Two events

Set notation review

Complement
(not)

Union (or)
(and)

Intersection
(and)

Difference (A, not B)

Symmetric difference (only one)

Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?

Ω = \{HH, HT, TH, TT\}; uniform probability space.

Event \(A = \) first flip is heads:
\(A = \) \{HH, HT\}.

New sample space: \(A\); uniform still.

Event \(B = \) two heads.
The probability of two heads if the first flip is heads.
The probability of \(B \) given \(A \) is 1/2.

A similar example.

Two coin flips. One of the flips is heads.
Probability of two heads?

Ω = \{HH, HT, TH, TT\}; uniform.

Event \(A = \) one flip is heads. \(A = \) \{HH, HT, TH\}.

New sample space: \(A\); uniform still.

Event \(B = \) two heads.
The probability of two heads if at least one flip is heads.
The probability of \(B \) given \(A \) is 1/3.

Conditional Probability: A non-uniform example

Consider \(Ω = \{1,2,\ldots,N\}\) with \(Pr[n] = p_n\).

\(Pr[3|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{p_3}{Pr[B]}\)

\(ω \not\in B \Rightarrow Pr[ω|B] = 0\).

Another non-uniform example

Consider \(Ω = \{1,2,\ldots,N\}\) with \(Pr[n] = p_n\).

\(Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}\).

Yet another non-uniform example

Consider \(Ω = \{1,2,\ldots,N\}\) with \(Pr[n] = p_n\).

\(Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}\).
**Conditional Probability.**

If in \( A \), what is the probability of outcome \( \omega \)?

If \( \omega \not\in A \), probability is 0

Otherwise: Ratio of probability of \( \omega \) to total probability of \( A \)

\[
Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}
\]

Uniform Probability Space:
Ratio of \( 1/|\Omega| \) to \( |A|/|\Omega| = \Rightarrow 1/|A| \).
(Makes sense!)

**Conditional Probability.**

**Definition:** The conditional probability of \( B \) given \( A \) is

\[
Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}
\]

From \( A \)!
In \( B \)?
Must be in \( A \cap B \).

\[
Pr[\omega|A] = \frac{Pr[A \cap \omega]}{Pr[A]}
\]

Note: \( Pr[A \cap \omega] = Pr[\omega] \times Pr[A|\omega] = Pr[A] \times Pr[\omega|A] \).

**Conditional Probability.**

**Definition:** The conditional probability of \( B \) given \( A \) is

\[
Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}
\]

You observe that the event \( B \) occurs.
That changes your information about the probability of every event \( A \).

The conditional probability of \( A \) given \( B \) is

\[
Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}
\]

Note: \( Pr[A \cap B] = Pr[B] \times Pr[A|B] = Pr[A] \times Pr[B|A] \).