What do we learn from observations?
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1. Probability Basics Review
2. Examples
3. Conditional Probability
Probability Basics Review

Setup:

1. \(\Pr(\omega) \leq 1\).
2. \(\sum_{\omega \in \Omega} \Pr(\omega) = 1\).

Event: \(A \subseteq \Omega\), \(\Pr(A) = \sum_{\omega \in \Omega} \Pr(\omega)\).

\(\Pr(\text{at least one } H \text{ out of two tosses}) = \Pr(HT, TH, HH) = \frac{3}{4}\)
Probability Basics Review

Setup:

▶ Random Experiment.
Setup:

- Random Experiment.
  
  Flip a coin twice.
Probability Basics Review

Setup:

- Random Experiment.
  
  Flip a coin twice.
  
- Probability Space.
Probability Basics Review

Setup:
- Random Experiment. 
  Flip a coin twice.
- Probability Space.
  - Sample Space: Set of outcomes, $\Omega$.
Probability Basics Review

Setup:

- Random Experiment.
  - Flip a coin twice.
- Probability Space.
  - **Sample Space:** Set of outcomes, $\Omega$.
    - $\Omega = \{HH, HT, TH, TT\}$
Probability Basics Review

Setup:

- Random Experiment.
  Flip a coin twice.
- Probability Space.
  - **Sample Space**: Set of outcomes, $\Omega$.
    $\Omega = \{HH, HT, TH, TT\}$
  - **Probability**: $Pr[\omega]$ for all $\omega \in \Omega$.
Probability Basics Review

Setup:

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    $Pr[HH] = \cdots = Pr[TT] = 1/4$
Probability Basics Review

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► Random Experiment.
  Flip a coin twice.

► Probability Space.
  ▶ **Sample Space:** Set of outcomes, \( \Omega \).
    \( \Omega = \{HH, HT, TH, TT\} \)
  ▶ **Probability:** \( Pr[\omega] \) for all \( \omega \in \Omega \).
    \( Pr[HH] = \cdots = Pr[TT] = 1/4 \)
    1. \( 0 \leq Pr[\omega] \leq 1 \).
Setup:

- Random Experiment.
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  - **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \leq Pr[\omega] \leq 1$.
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Probability Basics Review

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- **Random Experiment.**
  
  Flip a coin twice.

- **Probability Space.**
  
  - **Sample Space:** Set of outcomes, $\Omega$.
    
    $\Omega = \{HH, HT, TH, TT\}$

  - **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    
    $Pr[HH] = \ldots = Pr[TT] = 1/4$

    1. $0 \leq Pr[\omega] \leq 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1$. 

- **Event:** $A \subseteq \Omega$, $Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$. 

  $Pr[\text{at least one } H \text{ out of two tosses}] = Pr[HT, TH, HH] = 3/4$
Probability Basics Review

Setup:

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  Flip a coin twice.
- Probability Space.
  - **Sample Space:** Set of outcomes, \( \Omega \).
    \[ \Omega = \{HH, HT, TH, TT\} \]
  - **Probability:** \( Pr[\omega] \) for all \( \omega \in \Omega \).
    \[ Pr[HH] = \cdots = Pr[TT] = \frac{1}{4} \]
    1. \( 0 \leq Pr[\omega] \leq 1 \).
    2. \( \sum_{\omega \in \Omega} Pr[\omega] = 1 \).
  - **Event:** \( A \subseteq \Omega, Pr[A] = \sum_{\omega \in \Omega} Pr[\omega] \).
    \[ Pr[\text{at least one H out of two tosses}] = Pr[HT, TH, HH] = \frac{3}{4} \]
Probability Basics Review

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▷ Random Experiment.
   Flip a coin twice.

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   $\Omega = \{HH, HT, TH, TT\}$

▷ Probability: $Pr[\omega]$ for all $\omega \in \Omega$.
   $Pr[HH] = \cdots = Pr[TT] = 1/4$
   1. $0 \leq Pr[\omega] \leq 1$.
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▷ Event: $A \subseteq \Omega$, $Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$.
   $Pr[\text{at least one H out of two tosses}] = Pr[HT, TH, HH] = 3/4$
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses}$
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$. 
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$. 
$|\Omega| = 2 \times 2 \times \cdots \times 2$
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$.  
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$. 
Exactly 50 heads in 100 coin tosses.

Sample space: \( \Omega = \) set of 100 coin tosses = \( \{ H, T \}^{100} \).
\[ |\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}. \]

Uniform probability space: \( Pr[\omega] = \frac{1}{2^{100}}. \)
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$. 
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ “100 coin tosses with exactly 50 heads”
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$.  
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ “100 coin tosses with exactly 50 heads”  
$|E|$?  
Choose 50 positions out of 100 to be heads.
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.  
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ “100 coin tosses with exactly 50 heads” 

Choose 50 positions out of 100 to be heads. 

$|E| = \binom{100}{50}$. 
Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega = \text{set of 100 coin tosses} = \{H, T\}^{100}$. 
$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

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Event $E =$ “100 coin tosses with exactly 50 heads”
$|E|$?
Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$
Calculation.
Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$
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$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$ 

$$\left( \frac{2n}{n} \right) \approx \frac{\sqrt{4\pi n(2n/e)^{2n}}}{[\sqrt{2\pi n(n/e)^n}]^2}$$
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Stirling formula (for large $n$):

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n (2n/e)^{2n}}}{[\sqrt{2\pi n (n/e)^n}]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$
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$$Pr[E] \approx \frac{4^{50}}{\sqrt{50\pi}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$
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```ruby
irb(main):006:0> (50*3.1415926)**(0.5)
=> 12.5331412662588
irb(main):007:0> 1/12.5331412662588
=> 0.0797884567608089
```
Calculation.

Stirling formula (for large $n$):

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\end{align*}
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Exactly 50 heads in 100 coin tosses.

\[ Pr[n \text{Hs out of } 2n] = \frac{\binom{2n}{n}}{2^{2n}} \]
Probability of even number of heads in 57 coin tosses

\[ \Omega = \text{57 coin tosses}. \]

Let \( E = \{ \omega \in \Omega \mid \text{number of Hs in } \omega \text{ is even} \} \).

Fact: \( P(E) = \frac{1}{2} \).

Proof: Consider the correspondence:

\[ H A_1 A_2 \cdots A_{56} \leftrightarrow T A_1 A_2 \cdots A_{56} \]

Here, \( A_1, \ldots, A_{56} \) are 56 coin flips.

It matches every even sequence to an odd sequence, and conversely. Hence, there are exactly as many odd as even sequences.

\[ |E| = |\Omega| / 2 \]

\[ P(E) = \frac{|E|}{|\Omega|} = \frac{1}{2} \]
Probability of even number of heads in 57 coin tosses

\( \Omega = 57 \) coin tosses.

\[ \left| \Omega \right| = 2^{57} \]

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\[ \text{HA}_1 \text{A}_2 \cdots \text{A}_{56} \leftrightarrow \text{TA}_1 \text{A}_2 \cdots \text{A}_{56} \]

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\[ \Rightarrow \left| E \right| = \left| \Omega \right| / 2 \]

\[ \Rightarrow P(E) = \frac{\left| E \right|}{\left| \Omega \right|} = \frac{1}{2} \]
Probability of even number of heads in 57 coin tosses

\[ \Omega = 57 \text{ coin tosses.} \quad |\Omega| = 2^{57}. \]
Probability of even number of heads in 57 coin tosses

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It matches every even sequence to an odd sequence, and conversely.
Probability of even number of heads in 57 coin tosses

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Here, \( A_1, \ldots, A_{56} \) are 56 coin flips.

It matches every even sequence to an odd sequence, and conversely.

Hence, there are exactly as many odd as even sequences.
Probability of even number of heads in 57 coin tosses

$\Omega = 57$ coin tosses. $|\Omega| = 2^{57}$.

Let $E = \{\omega \in \Omega \mid$ number of Hs in $\omega$ is even$\}$. 

**Fact:** $P(E) = 1/2$.

**Proof:**
Consider the correspondence:

$$HA_1A_2\cdots A_{56} \leftrightarrow TA_1A_2\cdots A_{56}$$

Here, $A_1, \ldots, A_{56}$ are 56 coin flips.

It matches every even sequence to an odd sequence, and conversely.

Hence, there are exactly as many odd as even sequences.

$$\Rightarrow |E| = |\Omega|/2$$
Probability of even number of heads in 57 coin tosses

Ω = 57 coin tosses. |Ω| = 2^{57}.

Let \( E = \{ \omega \in \Omega \mid \text{number of Hs in } \omega \text{ is even} \} \).

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**Proof:**
Consider the correspondence:

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It matches every even sequence to an odd sequence, and conversely.

Hence, there are exactly as many odd as even sequences.

\[
\Rightarrow |E| = |\Omega|/2 \Rightarrow P(E) = \frac{|E|}{|\Omega|} = \frac{1}{2}
\]
Probability more heads than tails in 100 coin tosses.
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\[ \Omega = 100 \text{ coin tosses.} \]
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
\[ |\Omega| = 2^{100}. \]
Probability more heads than tails in 100 coin tosses.

\( \Omega = 100 \text{ coin tosses.} \)
|\( \Omega | = 2^{100}. \)

Recall event \( E = \text{‘equal heads and tails’} \)
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
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Recall event \( E = \) ‘equal heads and tails’
Event \( F = \) ‘more heads than tails’
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
\[ |\Omega| = 2^{100}. \]

Recall event \( E = \) ‘equal heads and tails’
Event \( F = \) ‘more heads than tails’
Event \( G = \) ‘more tails than heads’
Probability more heads than tails in 100 coin tosses.

Ω = 100 coin tosses.
|Ω| = 2^{100}.

Recall event $E$ = ‘equal heads and tails’
Event $F$ = ‘more heads than tails’
Event $G$ = ‘more tails than heads’
A 1-to-1 correspondence between outcomes in F and G!
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
\[ |\Omega| = 2^{100}. \]

Recall event \( E = \) ‘equal heads and tails’
Event \( F = \) ‘more heads than tails’
Event \( G = \) ‘more tails than heads’
A 1-to-1 correspondence between outcomes in \( F \) and \( G \)!
\[ |F| = |G|. \]
Probability more heads than tails in 100 coin tosses.

$\Omega = 100$ coin tosses.

$|\Omega| = 2^{100}$.

Recall event $E =$ ‘equal heads and tails’
Event $F =$ ‘more heads than tails’
Event $G =$ ‘more tails than heads’
A 1-to-1 correspondence between outcomes in $F$ and $G$!

$|F| = |G|$.

$E$, $F$ and $G$ are disjoint.
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
\[ |\Omega| = 2^{100}. \]

Recall event \( E = \) ‘equal heads and tails’
Event \( F = \) ‘more heads than tails’
Event \( G = \) ‘more tails than heads’
A 1-to-1 correspondence between
outcomes in \( F \) and \( G \)!
\[ |F| = |G|. \]
\( E, F \) and \( G \) are disjoint.
\[ Pr[E] \approx 8\%. \]
Probability more heads than tails in 100 coin tosses.

$\Omega = 100$ coin tosses.
$|\Omega| = 2^{100}$.

Recall event $E =$ ‘equal heads and tails’
Event $F =$ ‘more heads than tails’
Event $G =$ ‘more tails than heads’
A 1-to-1 correspondence between outcomes in $F$ and $G$!
$|F| = |G|$.
$E$, $F$ and $G$ are disjoint.
$Pr[E] \approx 8\%$. 
Probability more heads than tails in 100 coin tosses.

\( \Omega = 100 \) coin tosses.
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Event \( F = \) ‘more heads than tails’
Event \( G = \) ‘more tails than heads’
A 1-to-1 correspondence between outcomes in \( F \) and \( G \)!
\(|F| = |G|.
\( E, F \) and \( G \) are disjoint.
\( Pr[E] \approx 8\%.
\(|\Omega| = |E| + |F| + |G|. \)
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
\[ |\Omega| = 2^{100}. \]

Recall event \( E = \) ‘equal heads and tails’
Event \( F = \) ‘more heads than tails’
Event \( G = \) ‘more tails than heads’
A 1-to-1 correspondence between outcomes in \( F \) and \( G \)!
\[ |F| = |G|. \]
\( E, F \) and \( G \) are disjoint.
\( Pr[E] \approx 8\%. \)

\[ |\Omega| = |E| + |F| + |G|. \]
\[ \Rightarrow 1 = Pr[\Omega] = Pr[E] + 2Pr[F] \approx 8\% + 2Pr[F]. \]
Probability more heads than tails in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
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Recall event \( E = \) ‘equal heads and tails’
Event \( F = \) ‘more heads than tails’
Event \( G = \) ‘more tails than heads’
A 1-to-1 correspondence between outcomes in \( F \) and \( G \)!
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\[ |\Omega| = |E| + |F| + |G|. \]
\[ \Rightarrow 1 = Pr[\Omega] = Pr[E] + 2Pr[F] \approx 8\% + 2Pr[F]. \]

Solve for \(|F|\):
Probability more heads than tails in 100 coin tosses.

Ω = 100 coin tosses. 
|Ω| = 2^{100}.

Recall event $E$ = ‘equal heads and tails’
Event $F$ = ‘more heads than tails’
Event $G$ = ‘more tails than heads’
A 1-to-1 correspondence between outcomes in $F$ and $G$!

$|F| = |G|$.

$E$, $F$ and $G$ are disjoint.
$Pr[E] \approx 8\%$.

$|Ω| = |E| + |F| + |G|$.

$\Rightarrow 1 = Pr[Ω] = Pr[E] + 2Pr[F] \approx 8\% + 2Pr[F]$.

Solve for $|F|$:

$|F| \approx 46\%$
Probability of $n$ heads in 100 coin tosses.
Probability of $n$ heads in 100 coin tosses.

$\Omega = 100$ coin tosses.
Probability of \( n \) heads in 100 coin tosses.

\( \Omega = 100 \) coin tosses.

\( |\Omega| = 2^{100} \).
Probability of \( n \) heads in 100 coin tosses.

\[ \Omega = 100 \text{ coin tosses.} \]
\[ |\Omega| = 2^{100}. \]

Event \( E_n = \text{`}n \text{ heads'}; \]
Probability of $n$ heads in 100 coin tosses.

$\Omega = 100$ coin tosses.
$|\Omega| = 2^{100}.$

Event $E_n = \text{‘}n\text{ heads’}$; $|E_n| =$
Probability of \( n \) heads in 100 coin tosses.

\( \Omega = 100 \) coin tosses.
\( |\Omega| = 2^{100} \).

Event \( E_n = \text{‘} n \text{ heads’} \); \( |E_n| = \binom{100}{n} \).
Probability of $n$ heads in 100 coin tosses.

$\Omega = 100$ coin tosses.

$|\Omega| = 2^{100}$.

Event $E_n = \text{‘}n \text{ heads’}; |E_n| = \binom{100}{n}$
Probability of $n$ heads in 100 coin tosses.

$\Omega = 100$ coin tosses.
$|\Omega| = 2^{100}$.

Event $E_n = \text{‘}n\text{ heads’}$; $|E_n| = \binom{100}{n}$

$$p_n = Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$$
Roll a red and a blue die.
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]

Uniform: \( Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36} \) for all \( \omega \).
Roll a red and a blue die.

$$\Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2.$$  
Uniform: $Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36}$ for all $\omega$.

What is the probability of

1. the red die showing 6?
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]

Uniform: \( Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36} \) for all \( \omega \).

What is the probability of

1. the red die showing 6?
   \[ E_1 = \{(6, b) : 1 \leq b \leq 6\}, \]
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]

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What is the probability of

1. the red die showing 6?
   
   \[ E_1 = \{(6, b) : 1 \leq b \leq 6\}, \quad |E_1| = 6, \]

2. at least one die showing 6?
   
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   \( E = E_1 \cup E_2 = \text{red or blue die (or both) show 6} \)

\[ |E| = |E_1| + |E_2| - |E_1 \cap E_2| \]

\[ |E_1 \cap E_2| = \{(6, 6)\} \]

\[ |E| = 6 + 6 - 1 = 11 \]

\[ Pr[E] = \frac{11}{36} \]
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]

Uniform: \( Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36} \) for all \( \omega \).

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$$E_1 \cup E_2 = \{6, 6\}, |E_1 \cap E_2| = 1$$

$$Pr[E] = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|\Omega|} = \frac{6 + 6 - 1}{36} = \frac{11}{36}$$
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Roll a red and a blue die.

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   \[ |E| = 6 + 6 - 1 = 11 \]
   \[ Pr[E] = 11/36 \]
Roll a red and a blue die.
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$E_1 = \text{'Red die shows 6'}$;

$E_2 = \text{'Blue die shows 6'}$;

$E_1 \cup E_2 = \text{'At least one die shows 6'}$;

$\Pr[E_1] = \frac{1}{6}$,

$\Pr[E_2] = \frac{1}{6}$,

$\Pr[E_1 \cup E_2] = \frac{11}{36}$.

$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$
Roll a red and a blue die.

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Roll a red and a blue die.

\[\begin{align*}
|E_1 \cup E_2| &= |E_1| + |E_2| - |E_1 \cap E_2| \\
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E_1 \cup E_2 &= \text{‘At least one die shows 6’}
\end{align*}\]
Roll a red and a blue die.

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\[ Pr[E_1] = \frac{6}{36}, \quad Pr[E_2] = \frac{6}{36}, \quad Pr[E_1 \cup E_2] = \frac{11}{36}. \]
Inclusion/Exclusion

Note that,

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What is the probability of

1. the dice sum to 7?
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$$\Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2.$$ Uniform: $Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36}$ for all $\omega$.

What is the probability of

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   $$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\};$$
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   for each choice \( a \) of the value of the red die,
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   so there are 6 total choices.

2. the dice sum to 10?
Roll a red and a blue die.

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2. the dice sum to 10?
Roll a red and a blue die.

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   for each choice \( a \) of the value of the red die, there is exactly one choice \( b = 7 - a \) for the blue die, so there are 6 total choices.
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2. the dice sum to 10?
   \[ E = \{ (4, 6), (5, 5), (6, 4) \} \]
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]

Uniform: \( Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36} \) for all \( \omega \).

What is the probability of

1. the dice sum to 7?

\[ E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}; |E| = 6. \]

Counting argument:

for each choice \( a \) of the value of the red die,

there is exactly one choice \( b = 7 - a \) for the blue die,

so there are 6 total choices.

\[ Pr[E] = |E|/|\Omega| = 6/36 = 1/6. \]

2. the dice sum to 10?

\[ E = \{(4, 6), (5, 5), (6, 4)\} \]

\[ Pr[E] = |E|/|\Omega| \]
Roll a red and a blue die.

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   for each choice \( a \) of the value of the red die,
   there is exactly one choice \( b = 7 - a \) for the blue die,
   so there are 6 total choices.
   \( Pr[E] = |E|/|\Omega| = 6/36 = 1/6. \)

2. the dice sum to 10?
   \[ E = \{(4, 6), (5, 5), (6, 4)\} \]
   \( Pr[E] = |E|/|\Omega| = 3/36 \)
Roll a red and a blue die.

\[ \Omega = \{(a, b) : 1 \leq a, b \leq 6\} = \{1, 2, \ldots, 6\}^2. \]

Uniform: \( Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{36} \) for all \( \omega \).

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   \[ E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}; |E| = 6. \]
   Counting argument:
   for each choice \( a \) of the value of the red die, there is exactly one choice \( b = 7 - a \) for the blue die, so there are 6 total choices.
   \[ Pr[E] = \frac{|E|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}. \]

2. the dice sum to 10?
   \[ E = \{(4, 6), (5, 5), (6, 4)\} \]
   \[ Pr[E] = \frac{|E|}{|\Omega|} = \frac{3}{36} = \frac{1}{12}. \]
Roll a red and a blue die.

\[ Pr[\text{Sum to 7}] = \frac{6}{36} \quad \quad Pr[\text{Sum to 10}] = \frac{3}{36} \]
Roll two blue dice.
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The key idea is that we do not distinguish the dice. Roll die 1, then die 2. Then forget the order. For instance, we consider that (2, 5) and (5, 2) are the same outcome. We designate this outcome by (2, 5).
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Roll die 1, then die 2. Then forget the order.
For instance, we consider that \((2, 5)\) and \((5, 2)\) are the same outcome.
We designate this outcome by \((2, 5)\).
Thus,
\[
\Omega' = \{(a, b) \mid 1 \leq a \leq b \leq 6\}.
\]
Roll two blue dice.

The key idea is that we do not distinguish the dice.
Roll die 1, then die 2. Then forget the order.
For instance, we consider that $(2, 5)$ and $(5, 2)$ are the same outcome.
We designate this outcome by $(2, 5)$.
Thus,
\[ \Omega' = \{(a, b) \mid 1 \leq a \leq b \leq 6\}. \]
We see that $Pr[(1, 3)] =$
Roll two blue dice.

The key idea is that we do not distinguish the dice. Roll die 1, then die 2. Then forget the order. For instance, we consider that (2, 5) and (5, 2) are the same outcome. We designate this outcome by (2, 5). Thus,

$$\Omega' = \{(a, b) \mid 1 \leq a \leq b \leq 6\}.$$

We see that $Pr[(1, 3)] = \frac{2}{36}$ and $Pr[(2, 2)] =$
Roll two blue dice.

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We see that $Pr[(1, 3)] = \frac{2}{36}$ and $Pr[(2, 2)] = \frac{1}{36}$. 
Roll two blue dice

\[ \Omega : \text{Uniform} \]

\[ \Omega' : \text{Not uniform} \]

\[ P_r = \frac{2}{36} \]

\[ P_r = \frac{1}{36} \]
Roll two blue dice

Two different models of the same random experiment.
Roll two blue dice

Two different models of the same random experiment.

\[
\text{In } \Omega', Pr[(1,3)] = \frac{2}{36} \text{ and } Pr[(2,2)] = \frac{1}{36}
\]
Roll two blue dice.

Now what is the probability of at least one die showing 6?
Roll two blue dice.
Now what is the probability of at least one die showing 6?
Roll two blue dice.

Now what is the probability of at least one die showing 6?

In $\Omega$, $Pr[A] = \frac{11}{36}$;
Roll two blue dice.

Now what is the probability of at least one die showing 6?

In $\Omega$, $Pr[A] = \frac{11}{36}$; in $\Omega'$, $Pr[B] = 5 \times \frac{2}{36} + 1 \times \frac{1}{36}$. 
Roll two blue dice.

Now what is the probability of at least one die showing 6?

\[ \Omega : \text{Uniform} \]

\[ \Omega' : \text{Not uniform} \]

In \( \Omega \), \( \Pr[A] = \frac{11}{36} \); in \( \Omega' \), \( \Pr[B] = 5 \times \frac{2}{36} + 1 \times \frac{1}{36} \).

Of course, this is the same as for distinguishable dice!
Roll two blue dice.

Now what is the probability of at least one die showing 6?

In $\Omega$, $Pr[A] = \frac{11}{36}$; in $\Omega'$, $Pr[B] = 5 \times \frac{2}{36} + 1 \times \frac{1}{36}$.

Of course, this is the same as for distinguishable dice!

The event does not depend on the dice being distinguishable.
Roll two blue dice.

What is the probability of the dice sum to 7?
Roll two blue dice.
What is the probability of the dice sum to 7?

In $\Omega$, $\Pr[A] = \frac{6}{36}$; in $\Omega'$, $\Pr[B] = \frac{3 \times 2}{36}$.

Of course, this is the same as for distinguishable dice!
The event does not depend on the dice being distinguishable.
Roll two blue dice.

What is the probability of the dice sum to 7?

In $\Omega$, $Pr[A] = \frac{6}{36}$;
Roll two blue dice.
What is the probability of the dice sum to 7?

\[ \Omega : \text{Distinguishable dice} \]

\[ \Omega' : \text{Indistinguishable dice} \]

In \( \Omega \), \( Pr[A] = \frac{6}{36} \); in \( \Omega' \), \( Pr[B] = 3 \times \frac{2}{36} \).
Roll two blue dice.

What is the probability of the dice sum to 7?

In $\Omega$, $Pr[A] = \frac{6}{36}$; in $\Omega'$, $Pr[B] = 3 \times \frac{2}{36}$.

Of course, this is the same as for distinguishable dice!
Roll two blue dice.
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In $\Omega$, $Pr[A] = \frac{6}{36}$; in $\Omega'$, $Pr[B] = 3 \times \frac{2}{36}$.

Of course, this is the same as for distinguishable dice!
The event does not depend on the dice being distinguishable.
Really not uniform! and not finite!

Experiment: Toss three times a coin with $\Pr[H] = \frac{2}{3}$.

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

$\Pr[HHH] = \left(\frac{2}{3}\right)^3$;
$\Pr[HHT] = \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)$; . . .

Toss a fair coin until you get a heads.

$\Omega = \{H, TH, TTH, TTTH, . . .\}$

$\Pr[H] = \frac{1}{2}$, $\Pr[TH] = \frac{1}{4}$, $\Pr[TTH] = \frac{1}{8}$

Still sums to 1. Indeed $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$. 
Really not uniform! and not finite!

- Experiment: Toss three times a coin with \( Pr[H] = \frac{2}{3} \).
Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = 2/3$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. 
- Toss a fair coin until you get a heads.
  - $\Omega = \{H, TH, TTH, TTTH, \ldots\}$.
  - $Pr[H] = 1/2$, $Pr[TH] = 1/4$, $Pr[TTH] = 1/8$, $Pr[TTTH] = 1/16$, $\ldots$
  - Still sums to 1. Indeed $1/2 + 1/4 + 1/8 + 1/16 + \ldots = 1$. 

Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = 2/3$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $Pr[HHH] = \left(\frac{2}{3}\right)^3$;
Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = 2/3$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $Pr[HHH] = (\frac{2}{3})^3$; $Pr[HHT] = (\frac{2}{3})^2(\frac{1}{3})$; ...
Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = 2/3$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $Pr[HHH] = \left(\frac{2}{3}\right)^3$; $Pr[HHT] = \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)$; ... 
- Toss a fair coin until you get a heads.
Really not uniform! and not finite!

- Experiment: Toss three times a coin with \( Pr[H] = 2/3 \).
  - \( \Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \).
  - \( Pr[HHH] = (\frac{2}{3})^3; \) \( Pr[HHT] = (\frac{2}{3})^2(\frac{1}{3}); \ldots \)
- Toss a fair coin until you get a heads.
  - \( \Omega = \{H, TH, TTH, TTTH, \ldots\} \)
Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = \frac{2}{3}$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $Pr[HHH] = \left(\frac{2}{3}\right)^3; \ Pr[HHT] = \left(\frac{2}{3}\right)^2(\frac{1}{3}); \ldots$

- Toss a fair coin until you get a heads.
  - $\Omega = \{H, TH, TTH, TTTH, \ldots\}$
  - $Pr[H] = \frac{1}{2}$,
Really not uniform! and not finite!

- Experiment: Toss three times a coin with $Pr[H] = \frac{2}{3}$.
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
  - $Pr[HHH] = \left(\frac{2}{3}\right)^3$; $Pr[HHT] = \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)$; …

- Toss a fair coin until you get a heads.
  - $\Omega = \{H, TH, TTH, TTTTH, \ldots\}$
  - $Pr[H] = \frac{1}{2}$, $Pr[TH] = \frac{1}{4}$,
Really not uniform! and not finite!

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- Toss a fair coin until you get a heads.
  - $\Omega = \{H, TH, TTH, TTTH, \ldots\}$
  - $Pr[H] = \frac{1}{2}, Pr[TH] = \frac{1}{4}, Pr[TTH] = \frac{1}{8}$
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- Toss a fair coin until you get a heads.
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  - $Pr[H] = \frac{1}{2}$, $Pr[TH] = \frac{1}{4}$, $Pr[TTH] = \frac{1}{8}$
  - Still sums to 1. Indeed $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$. 
Set notation review

\[ \Omega \]

\[ A \cap B \]

\[ \neg A \]

\[ A \cup B \]

\[ A \setminus B \]

\[ A \Delta B \]

**Figure**: Two events
Set notation review

Figure: Two events

Figure: Complement (not)
Set notation review

Figure: Two events

Figure: Union (or)

Figure: Complement (not)
Set notation review

Figure: Two events

Figure: Union (or)

Figure: Complement (not)

Figure: Intersection (and)
Set notation review

Figure: Two events

Figure: Union (or)

Figure: Difference (A, not B)

Figure: Complement (not)

Figure: Intersection (and)
Set notation review

Figure: Two events

Figure: Union (or)

Figure: Difference (A, not B)

Figure: Complement (not)

Figure: Intersection (and)

Figure: Symmetric difference (only one)
Conditional probability: example.

Two coin flips.
Conditional probability: example.

Two coin flips. First flip is heads.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?

Ω = \{HH, HT, TH, TT\};
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads?
\( \Omega = \{ HH, HT, TH, TT \} \); Uniform probability space.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A = \) first flip is heads:
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? 
\( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. 
Event \( A = \) first flip is heads: \( A = \{HH, HT\} \).
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? $\Omega = \{ HH, HT, TH, TT \}$; Uniform probability space. Event $A =$ first flip is heads: $A = \{ HH, HT \}$.

![Probability Space Diagram](image)
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{ HH, HT, TH, TT \} \); Uniform probability space. Event \( A = \) first flip is heads: \( A = \{ HH, HT \} \).

New sample space: \( A \);
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? $\Omega = \{HH, HT, TH, TT\}$; Uniform probability space. 
Event $A =$ first flip is heads: $A = \{HH, HT\}$.

New sample space: $A$; uniform still.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A \) = first flip is heads: \( A = \{HH, HT\} \).

New sample space: \( A \); uniform still.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{ HH, HT, TH, TT \} \); Uniform probability space.
Event \( A = \) first flip is heads: \( A = \{ HH, HT \} \).

New sample space: \( A \); uniform still.

Event \( B = \) two heads.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? \( \Omega = \{HH, HT, TH, TT\} \); Uniform probability space. Event \( A \) = first flip is heads: \( A = \{HH, HT\} \).

New sample space: \( A \); uniform still.

Event \( B \) = two heads.

The probability of two heads if the first flip is heads.
Conditional probability: example.

Two coin flips. First flip is heads. Probability of two heads? $\Omega = \{HH, HT, TH, TT\}$; Uniform probability space.
Event $A =$ first flip is heads: $A = \{HH, HT\}$.

New sample space: $A$; uniform still.

Event $B =$ two heads.

The probability of two heads if the first flip is heads. **The probability of $B$ given $A$**
Two coin flips. First flip is heads. Probability of two heads?
\( \Omega = \{ HH, HT, TH, TT \} \); Uniform probability space.
Event \( A = \) first flip is heads: \( A = \{ HH, HT \} \).

New sample space: \( A \); uniform still.

Event \( B = \) two heads.

The probability of two heads if the first flip is heads. **The probability of \( B \) given \( A \) is 1/2.**
A similar example.

Two coin flips.
A similar example.

Two coin flips. One of the flips is heads.
A similar example.

Two coin flips. One of the flips is heads.
Probability of two heads?
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \}; \]
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \} ; \text{ uniform}. \]
A similar example.

Two coin flips. One of the flips is heads.
Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \} \]; uniform.
Event \( A = \) one flip is heads.
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \}; \text{ uniform.} \]

Event \( A \) = one flip is heads. \( A = \{ HH, HT, TH \} \).
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \}; \text{ uniform.} \]

Event \( A = \) one flip is heads. \( A = \{ HH, HT, TH \}. \)
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Ω = \{HH, HT, TH, TT\}; uniform.

Event \(A = \text{one flip is heads.} \) \(A = \{HH, HT, TH\}\).

New sample space: \(A\);
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

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New sample space: \( A \); uniform still.
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

$$\Omega = \{HH, HT, TH, TT\}; \text{ uniform.}$$

Event $$A = \text{one flip is heads.} \ A = \{HH, HT, TH\}.$$ 

New sample space: $$A; \text{ uniform still.}$$
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

Ω = \{HH, HT, TH, TT\}; uniform.
Event A = one flip is heads. A = \{HH, HT, TH\}.

New sample space: A; uniform still.

Event B = two heads.
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \}; \text{ uniform.} \]
Event \( A \) = one flip is heads. \( A = \{ HH, HT, TH \} \).

New sample space: \( A \); uniform still.

Event \( B \) = two heads.

The probability of two heads if at least one flip is heads.
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{ HH, HT, TH, TT \}; \text{ uniform.} \]
Event \( A = \) one flip is heads. \( A = \{ HH, HT, TH \}. \)

New sample space: \( A; \) uniform still.

Event \( B = \) two heads.

The probability of two heads if at least one flip is heads.  
**The probability of \( B \) given \( A \)**
A similar example.

Two coin flips. One of the flips is heads. Probability of two heads?

\[ \Omega = \{HH, HT, TH, TT\} \]; uniform.
Event \( A = \) one flip is heads. \( A = \{HH, HT, TH\} \).

New sample space: \( A \); uniform still.

Event \( B = \) two heads.
The probability of two heads if at least one flip is heads. The probability of \( B \) given \( A \) is \( \frac{1}{3} \).
Conditional Probability: A non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$. 


$\omega / B \Rightarrow Pr[\omega | B] = 0$. 
Conditional Probability: A non-uniform example

Consider \( \Omega = \{1, 2, \ldots, N\} \) with \( Pr[n] = p_n \).
Conditional Probability: A non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

$Pr[3|B] =$
Conditional Probability: A non-uniform example

Consider \( \Omega = \{1, 2, \ldots, N\} \) with \( Pr[n] = p_n \).

\[
Pr[3|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{p_3}{Pr[B]}.
\]
Conditional Probability: A non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

\[
Pr[3|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{p_3}{Pr[B]}.
\]

$\omega \notin B \Rightarrow Pr[\omega|B] = \ldots$
Conditional Probability: A non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

\[
Pr[3|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{p_3}{Pr[B]}.
\]

$\omega \notin B \Rightarrow Pr[\omega|B] = 0$. 

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$. 
Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$. 
Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

$Pr[A|B] =$
Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$
Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$. 
Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$. 
Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}.$$
Yet another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$
Conditional Probability.

If in $A$, what is the probability of outcome $\omega$?

If $\omega \not\in A$, probability is 0.

Otherwise: Ratio of probability of $\omega$ to total probability of $A$

$$Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}$$

Uniform Probability Space:

Ratio of $1/|\Omega|$ to $|A|/|\Omega| = \frac{1}{|A|}$.

(Makes sense!)
Conditional Probability.

If in \( A \), what is the probability of outcome \( \omega \)?
If \( \omega \not\in A \),

\[
Pr[\omega|A] =? 
\]
Conditional Probability.

If in $A$, what is the probability of outcome $\omega$?
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Conditional Probability.

If in $A$, what is the probability of outcome $\omega$?

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Otherwise: Ratio of probability of $\omega$ to total probability of $A$

$$Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}$$

Uniform Probability Space:

$$= \frac{1}{|\Omega|} \cdot \frac{|A|}{|\Omega|} = \frac{1}{|A|}.$$

(Makes sense!)
Conditional Probability.

If in \( A \), what is the probability of outcome \( \omega \)?

If \( \omega \notin A \), probability is 0

Otherwise: Ratio of probability of \( \omega \) to total probability of \( A \)

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Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}
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Uniform Probability Space:
Conditional Probability.

If in $A$, what is the probability of outcome $\omega$?
If $\omega \notin A$, probability is 0
Otherwise: Ratio of probability of $\omega$ to total probability of $A$

$$Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}$$

Uniform Probability Space:
Ratio of $1/|\Omega|$ to $|A|/|\Omega|$ $\implies$ $1/|A|.$
Conditional Probability.

If in $A$, what is the probability of outcome $\omega$?

If $\omega \notin A$, probability is 0

Otherwise: Ratio of probability of $\omega$ to total probability of $A$

$$Pr[\omega|A] = \frac{Pr[\omega]}{Pr[A]}$$

Uniform Probability Space:

Ratio of $1/|\Omega|$ to $|A|/|\Omega| \implies 1/|A|$. (Makes sense!)
Definition: The conditional probability of $B$ given $A$ is

$$Pr[B|A] = \sum_{\omega \in B} Pr[\omega|A] = \frac{\sum_{\omega \in A \cap B} Pr[\omega]}{Pr[A]}$$
Definition: The conditional probability of $B$ given $A$ is

$$Pr[B|A] = \sum_{\omega \in B} Pr[\omega|A] = \frac{\sum_{\omega \in A \cap B} Pr[\omega]}{Pr[A]} = \frac{Pr[A \cap B]}{Pr[A]}$$
Definition: The conditional probability of $B$ given $A$ is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$
Conditional Probability.

**Definition:** The **conditional probability** of \( B \) given \( A \) is

\[
Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}
\]

\( \text{In } A! \)
Definition: The conditional probability of $B$ given $A$ is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$
**Definition:** The conditional probability of $B$ given $A$ is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$

- In $A$!
- In $B$?
- Must be in $A \cap B$. 

**Conditional Probability.**
Definition: The conditional probability of $B$ given $A$ is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$

In $A!$

In $B$?

Must be in $A \cap B$. 

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$
What do we learn from observations?

You observe that the event $B$ occurs. That changes your information about the probability of every event $A$. The conditional probability of $A$ given $B$ is

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}.$$

Note: $\Pr[A \cap B] = \Pr[B] \times \Pr[A \mid B] = \Pr[A] \times \Pr[B \mid A]$. 
What do we learn from observations?

You observe that the event $B$ occurs.
What do we learn from observations?

You observe that the event $B$ occurs.
That changes your information about the probability of every event $A$. 

\[
Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}
\]

Note: 
\[
Pr[A \cap B] = Pr[B] \times Pr[A \mid B] = Pr[A] \times Pr[B \mid A]
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What do we learn from observations?

You observe that the event $B$ occurs.
That changes your information about the probability of every event $A$.
The **conditional probability of $A$ given $B$ is**
You observe that the event $B$ occurs.
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The conditional probability of $A$ given $B$ is

$$Pr[A|B] =$$
What do we learn from observations?

You observe that the event $B$ occurs.

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The **conditional probability of $A$ given $B$** is

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}.$$
What do we learn from observations?

You observe that the event $B$ occurs.
That changes your information about the probability of every event $A$.
The conditional probability of $A$ given $B$ is

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}.$$  

Note: $Pr[A \cap B] = Pr[B] \times \ldots$
What do we learn from observations?

You observe that the event $B$ occurs.
That changes your information about the probability of every event $A$.

The conditional probability of $A$ given $B$ is

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Note: $Pr[A \cap B] = Pr[B] \times Pr[A|B]$
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You observe that the event $B$ occurs.

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Note: $Pr[A \cap B] = Pr[B] \times Pr[A|B] = Pr[A] \times$
You observe that the event $B$ occurs.

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