CS70: Jean Walrand: Lecture 23.

What do we learn from observations?

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1. Probability Basics Review
2. Examples
3. Conditional Probability

Probability Basics Review

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- Random Experiment.

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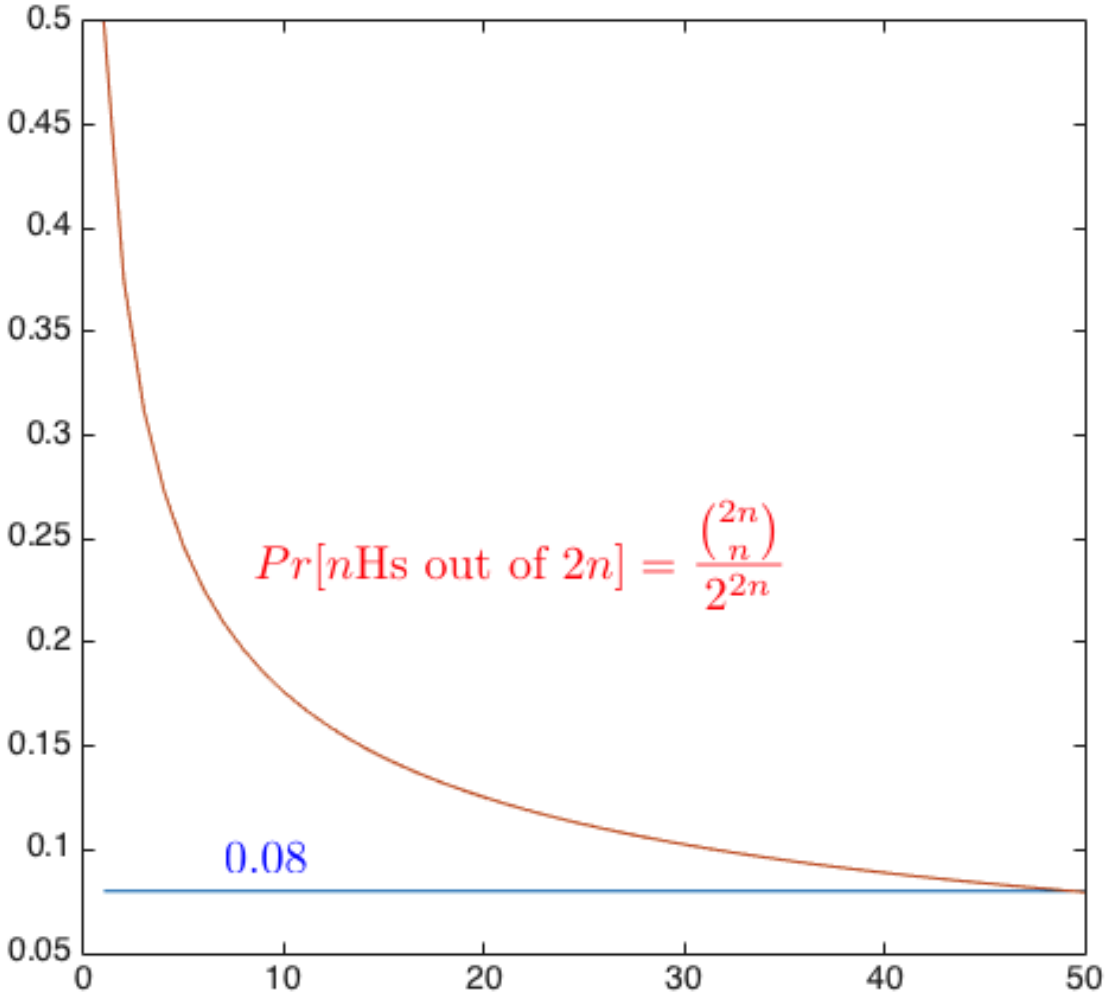

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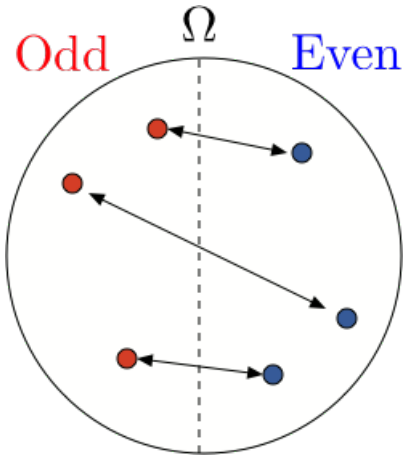

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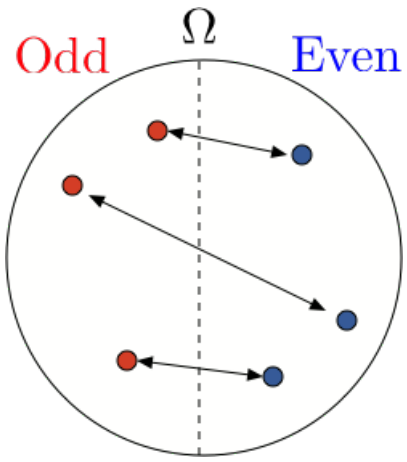

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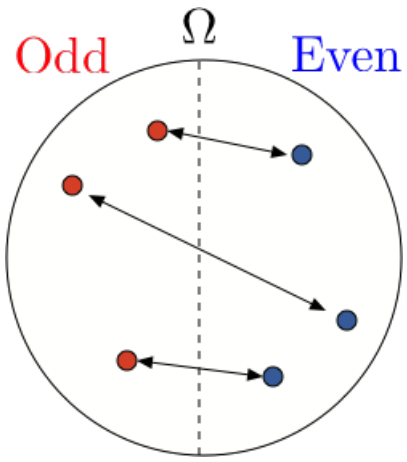

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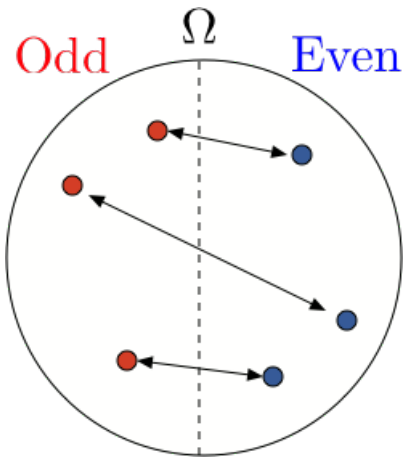

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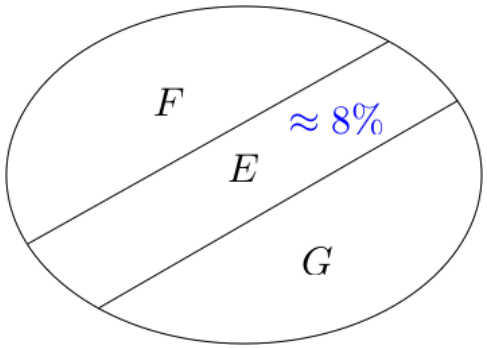

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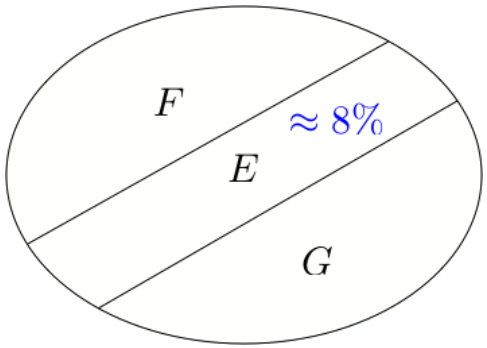

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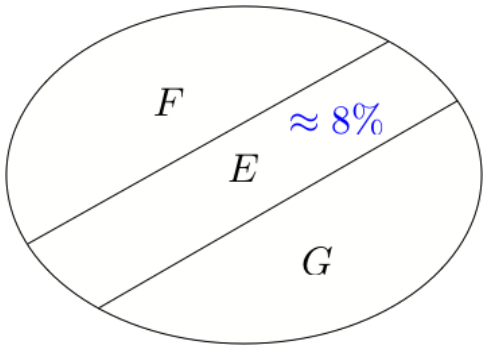

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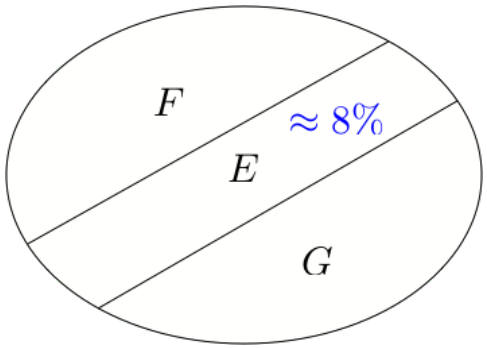

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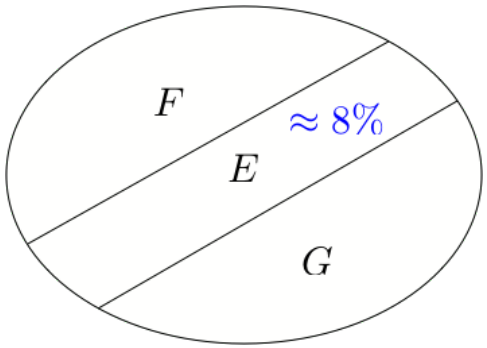

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$$|F| \approx 46\%$$

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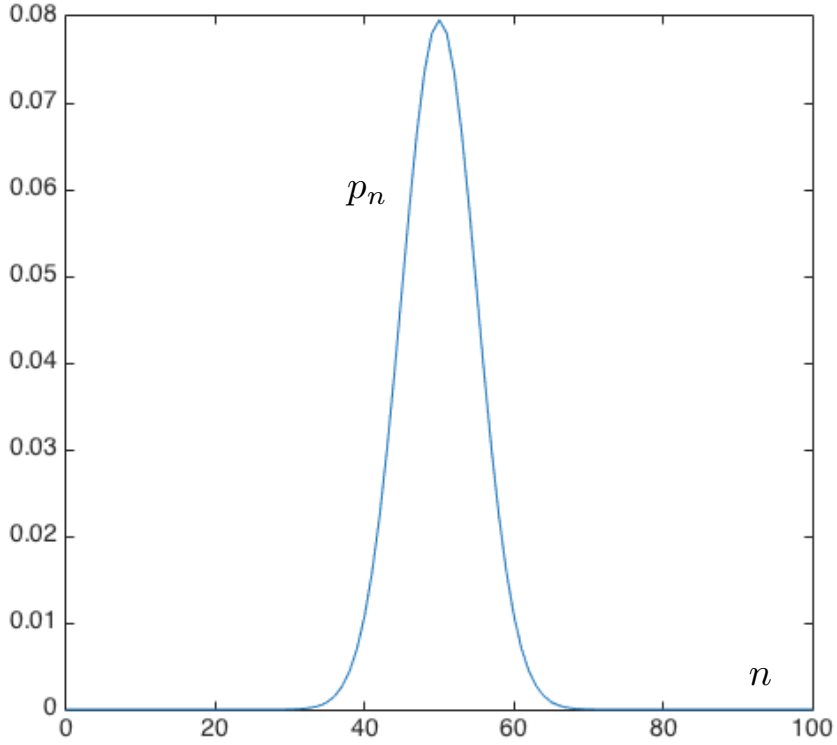

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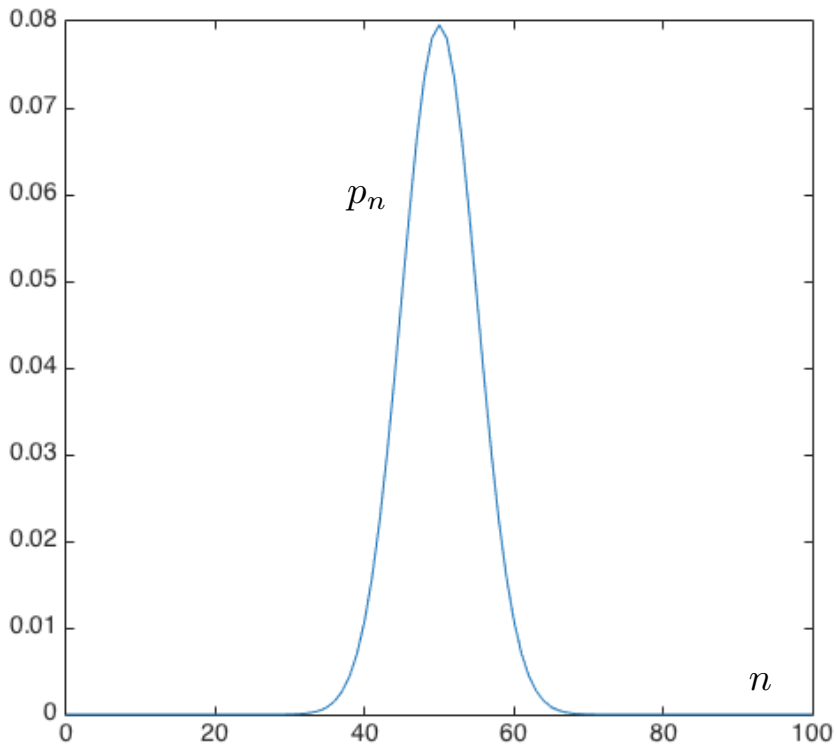

$$p_n = Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$$

Roll a red and a blue die.

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What is the probability of

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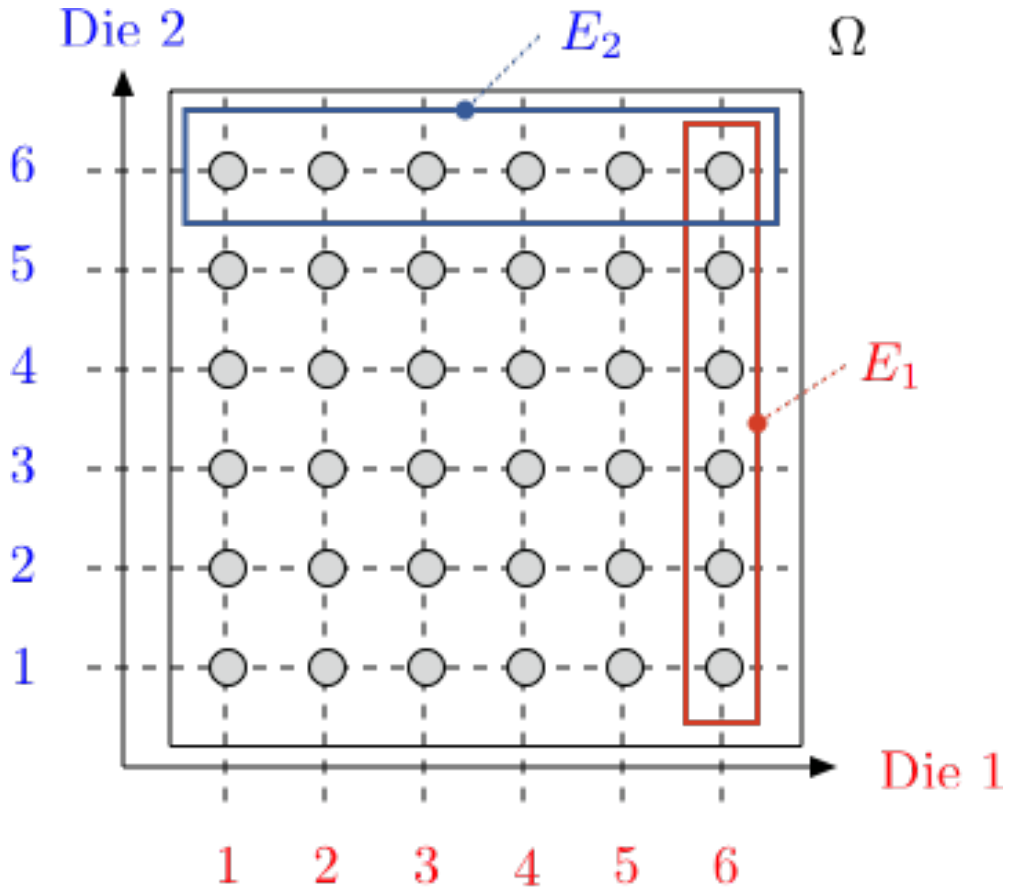

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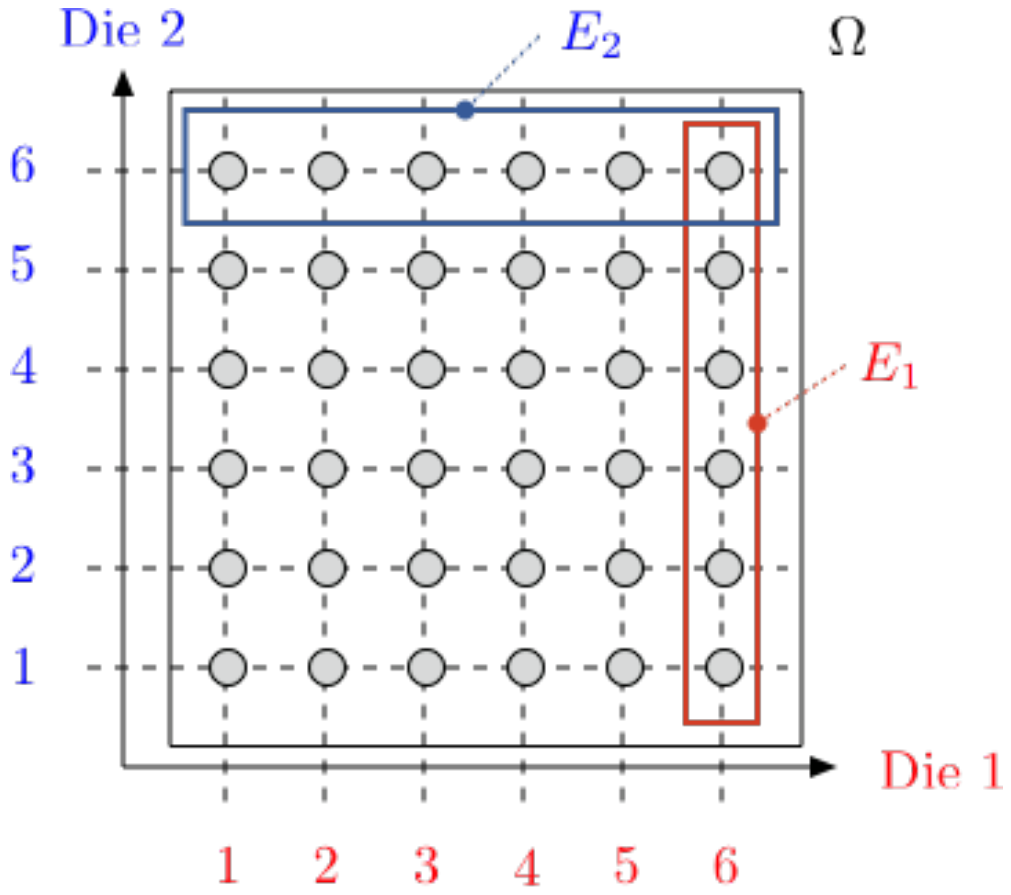

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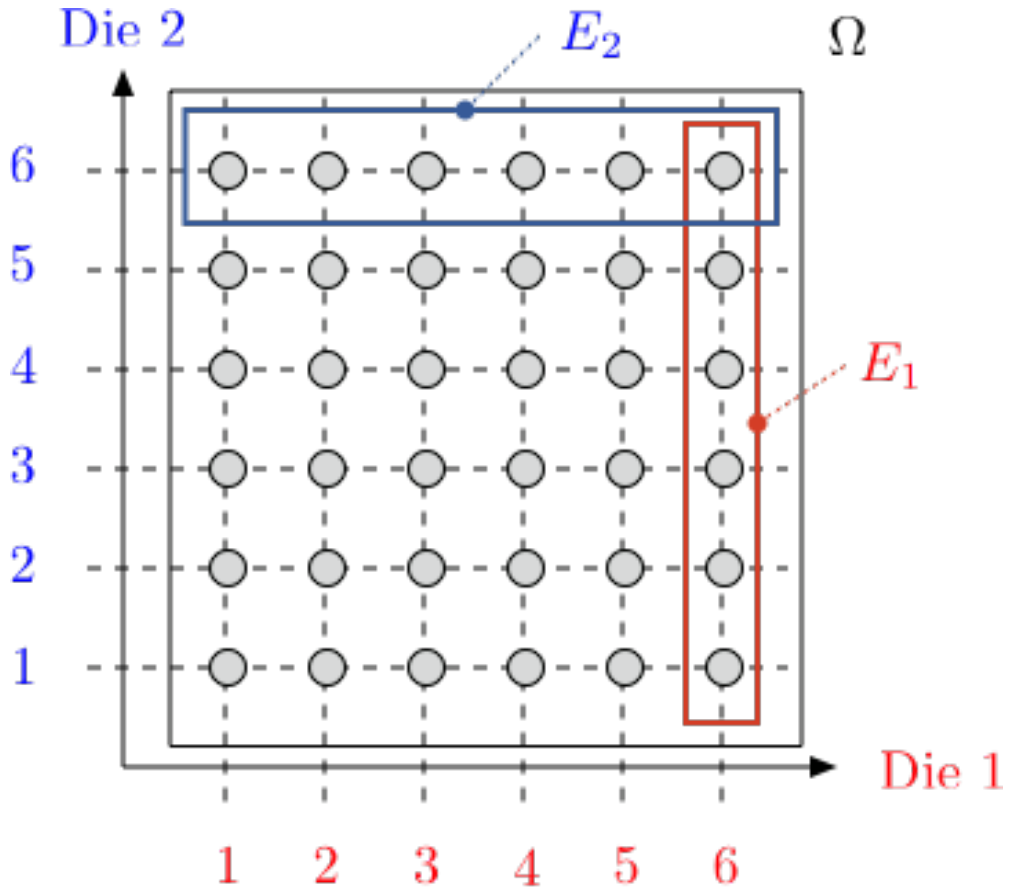

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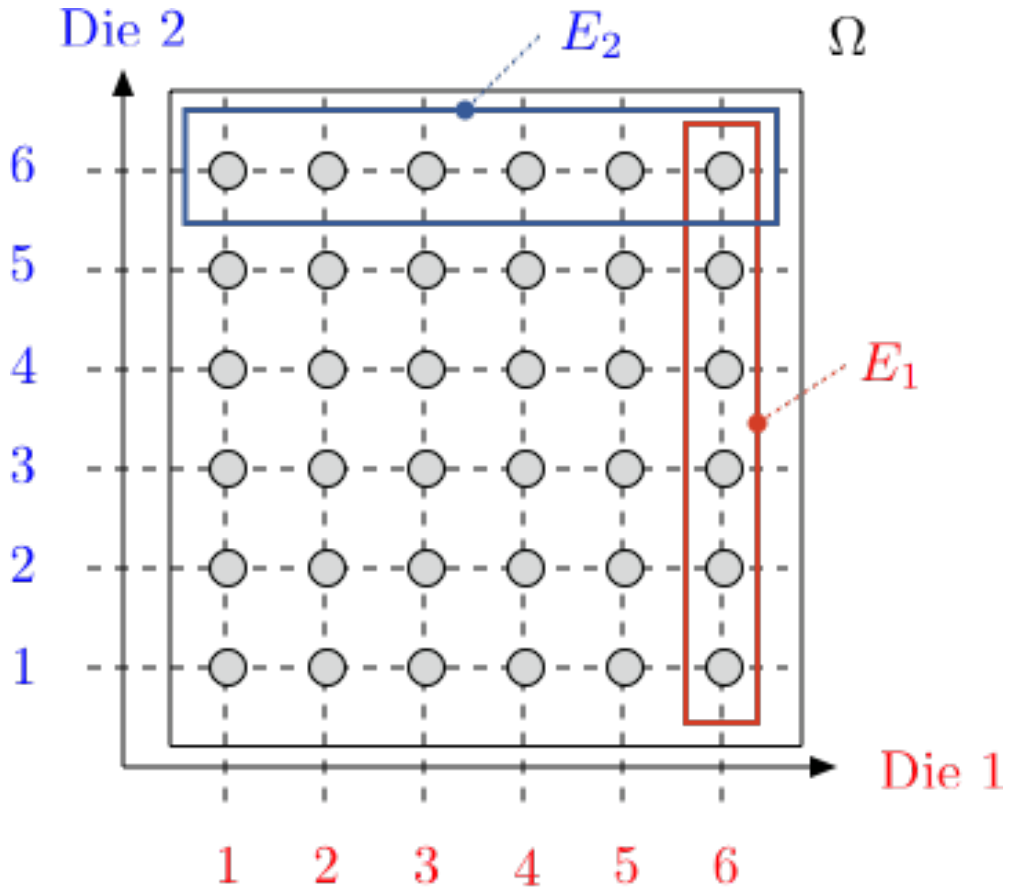

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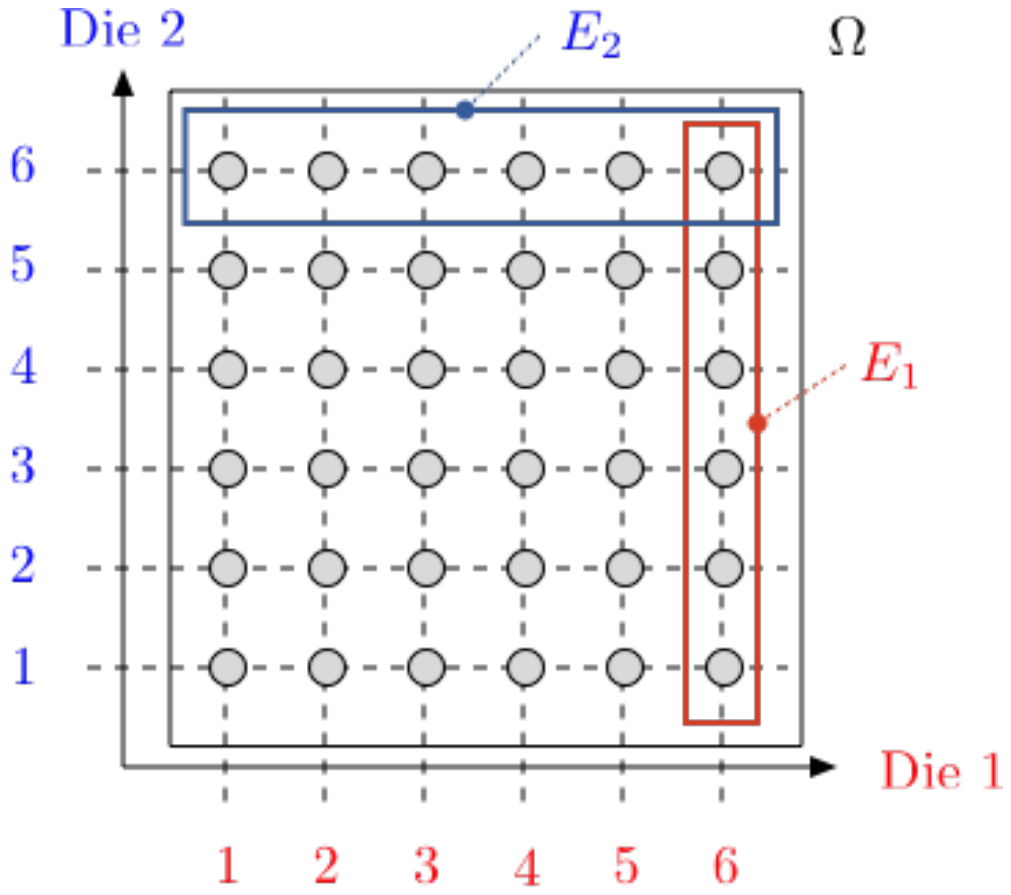

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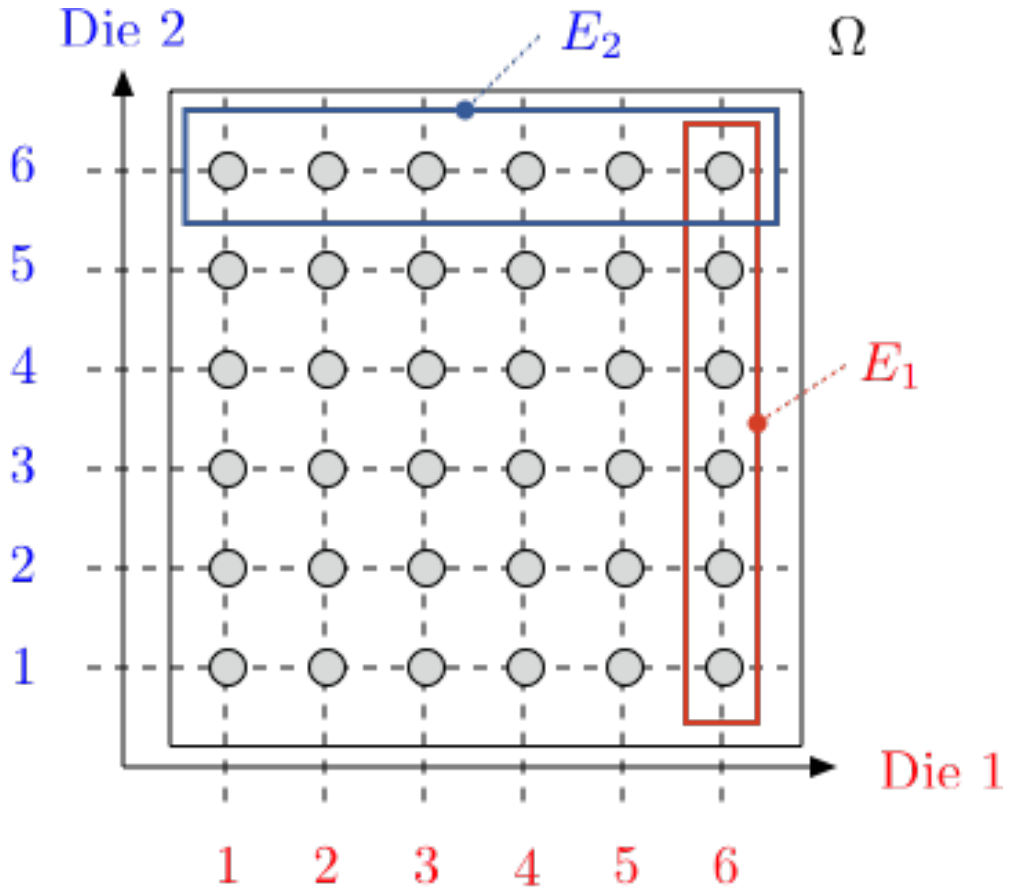

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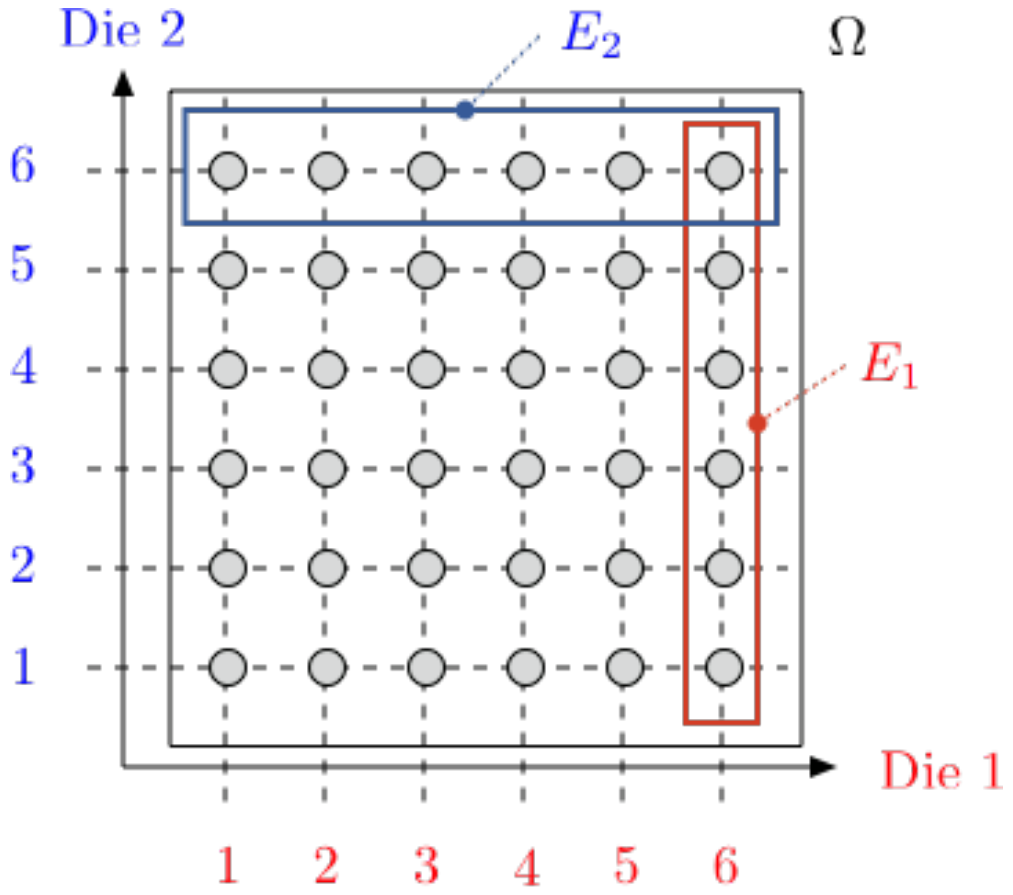

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Inclusion/Exclusion

Note that,

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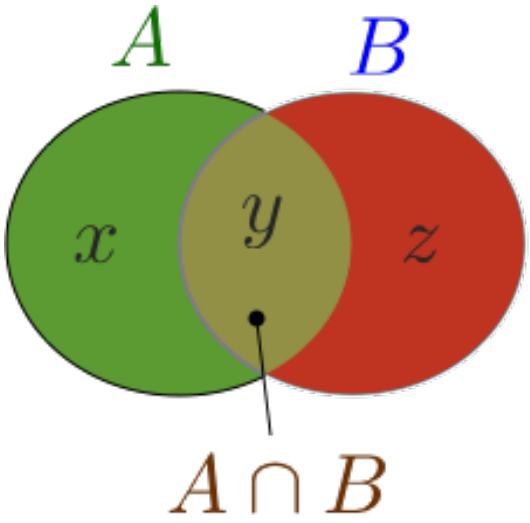

$$\begin{aligned} Pr[A] &= x + y \\ Pr[B] &= y + z \\ Pr[A \cap B] &= y \\ Pr[A \cup B] &= x + y + z \end{aligned}$$

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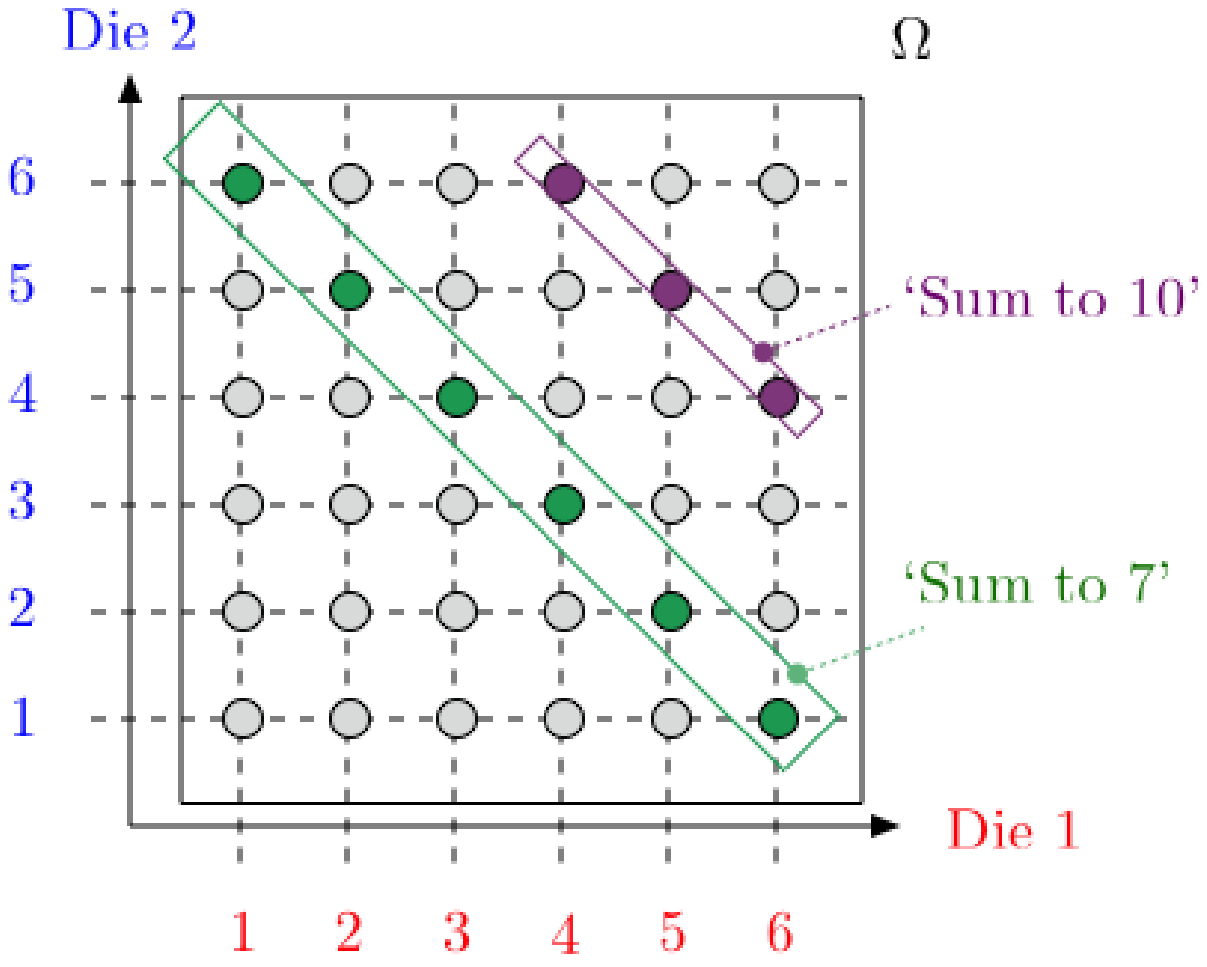

$$Pr[\text{Sum to } 7] = \frac{6}{36} \qquad Pr[\text{Sum to } 10] = \frac{3}{36}$$

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The key idea is that we do not distinguish the dice.

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Roll die 1, then die 2. Then forget the order.

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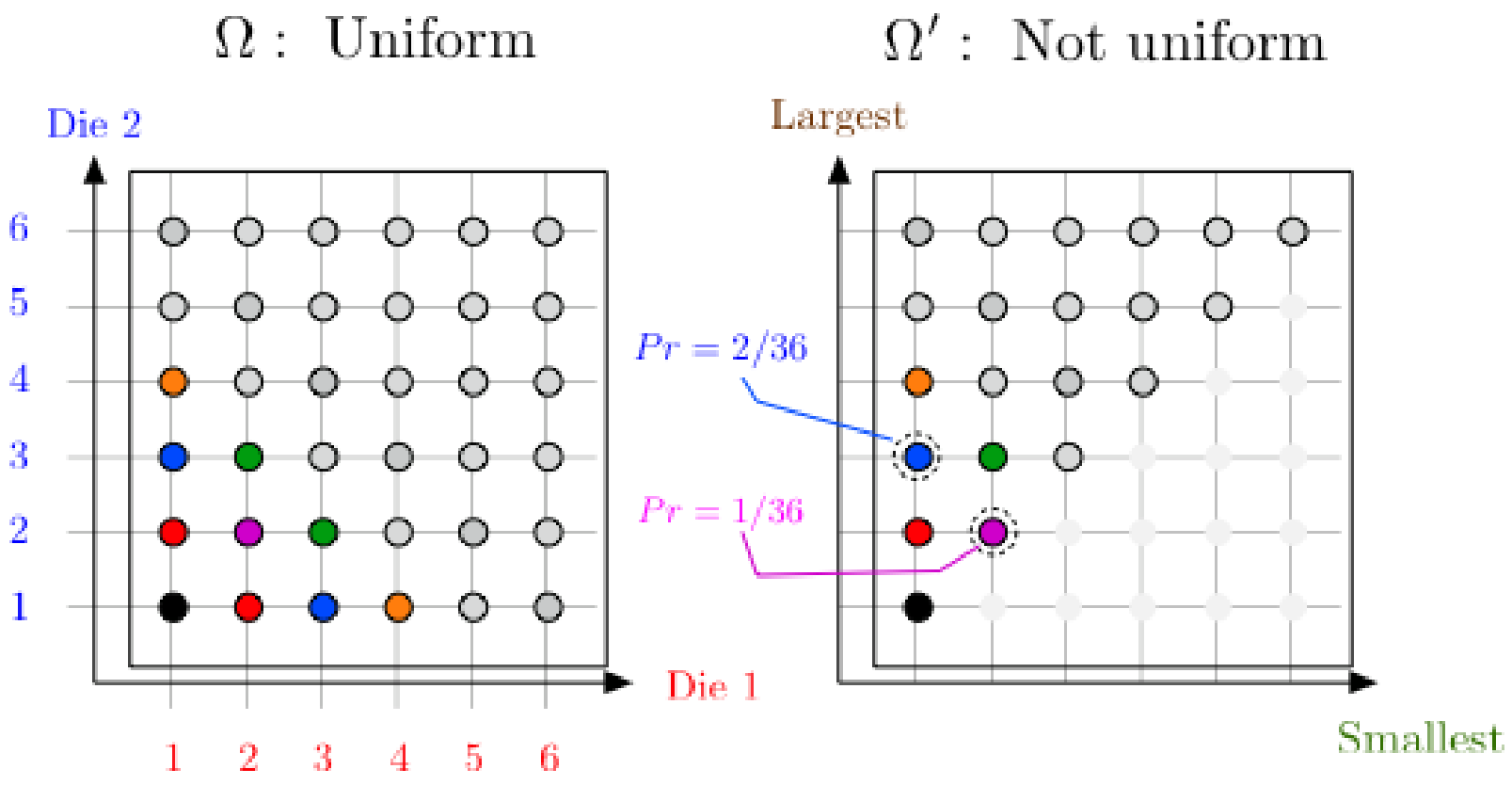

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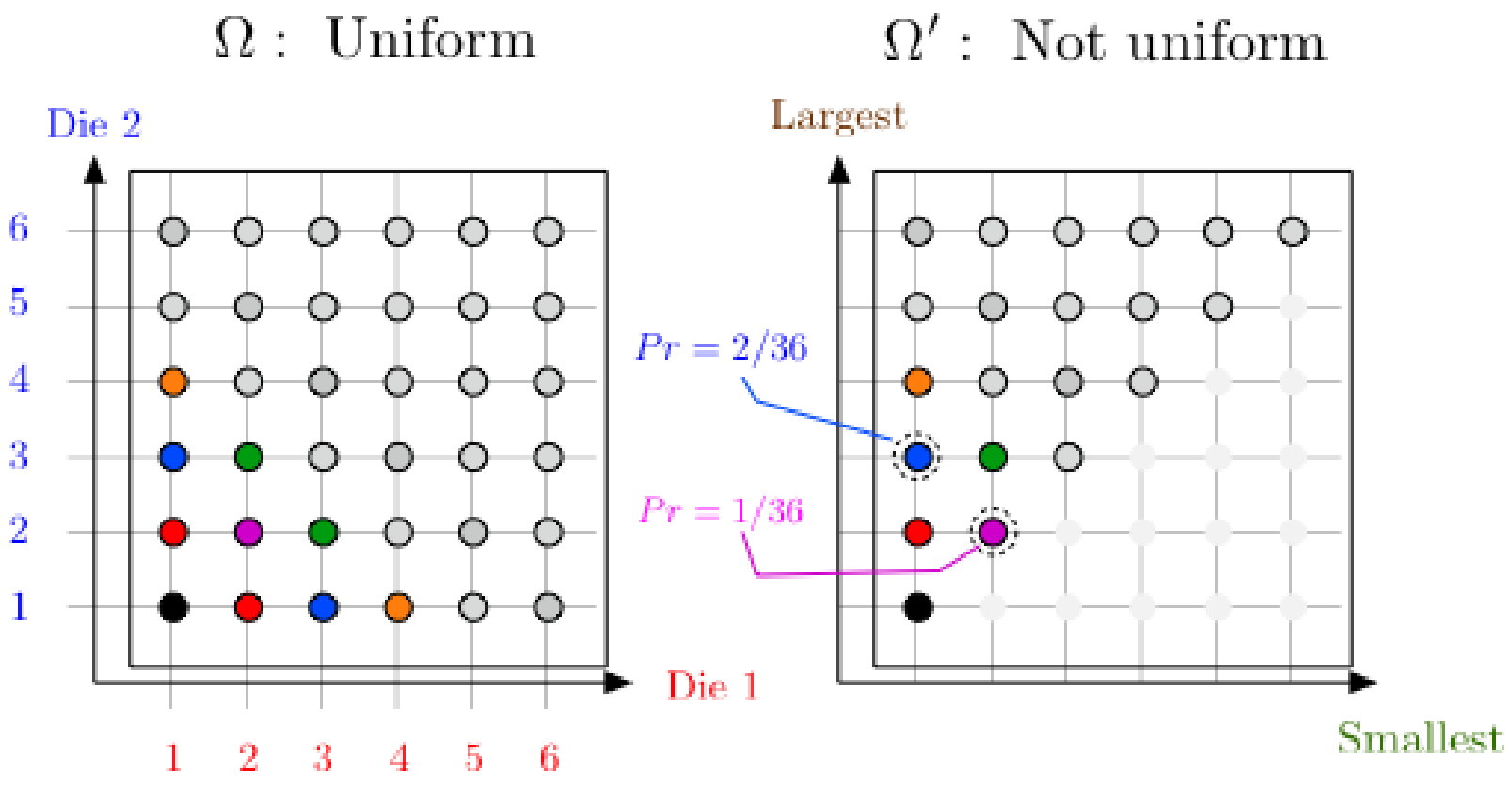

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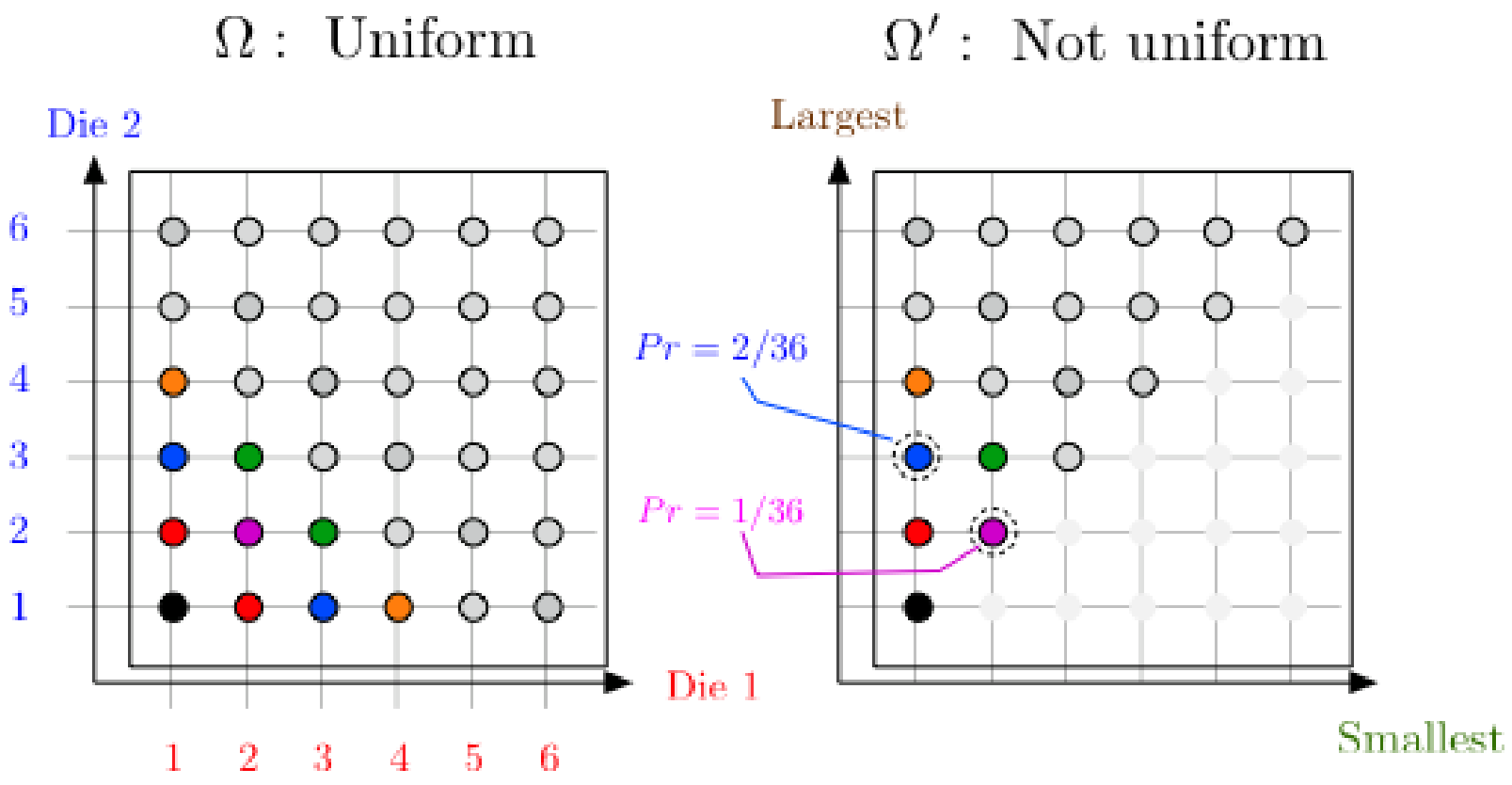

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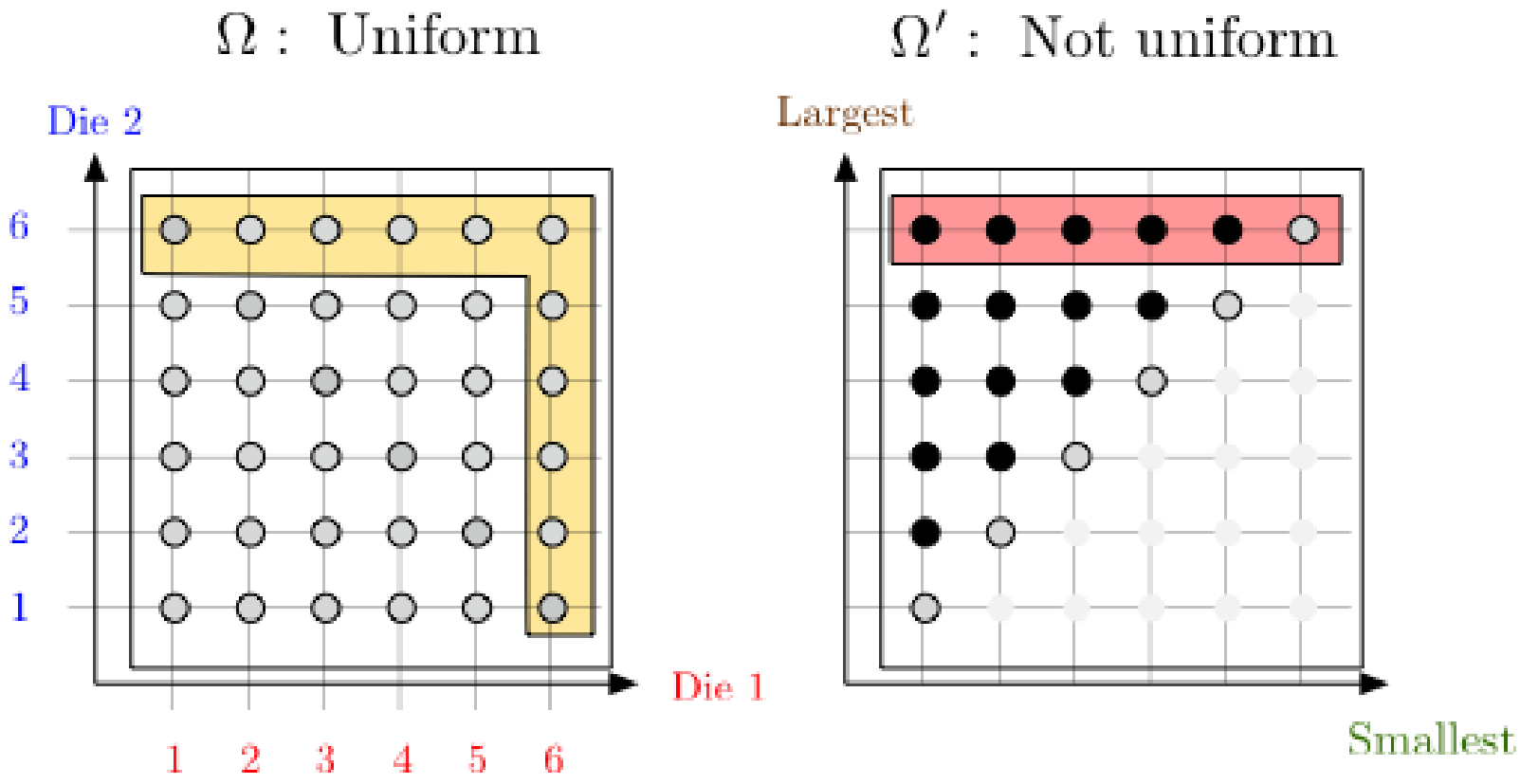

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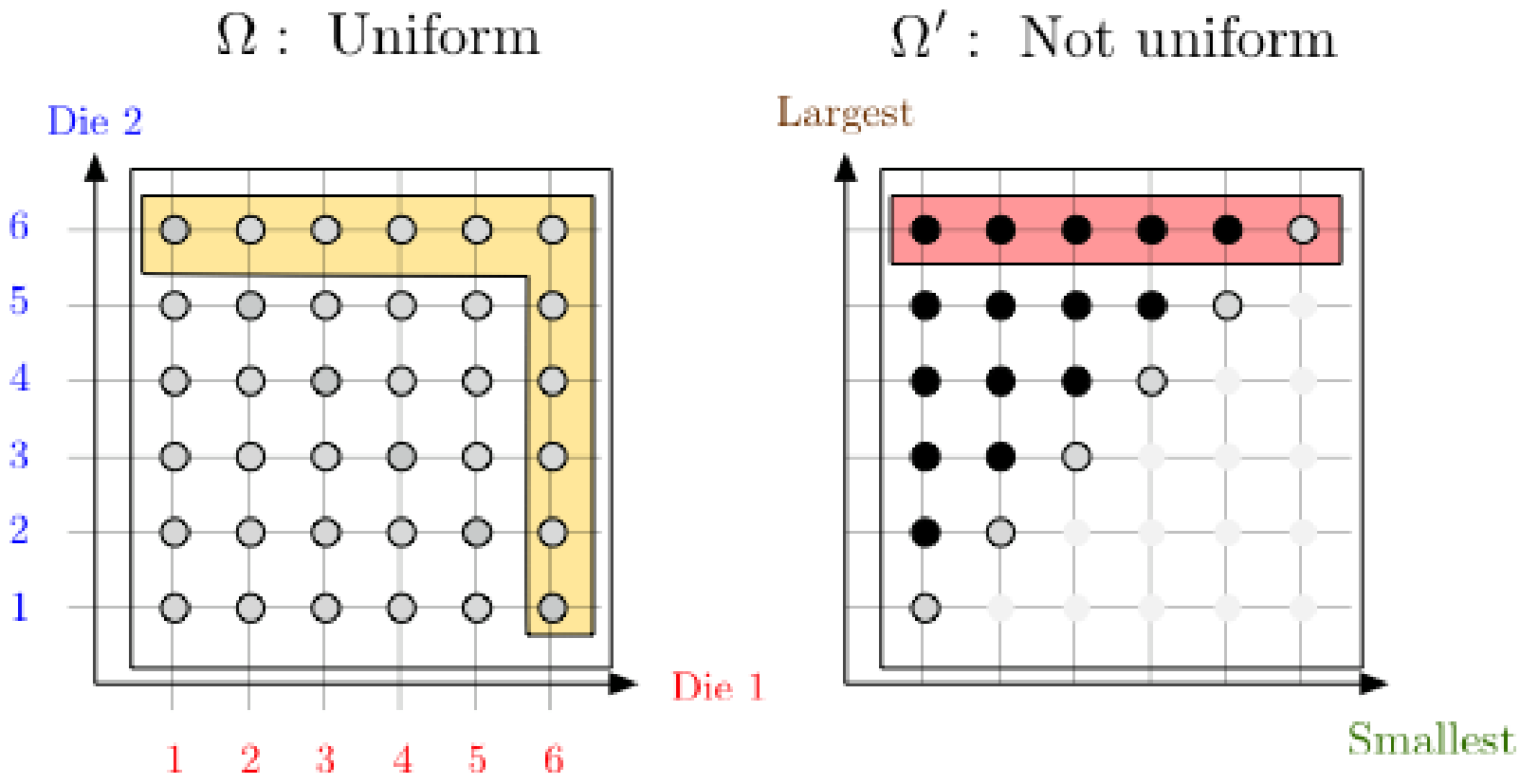

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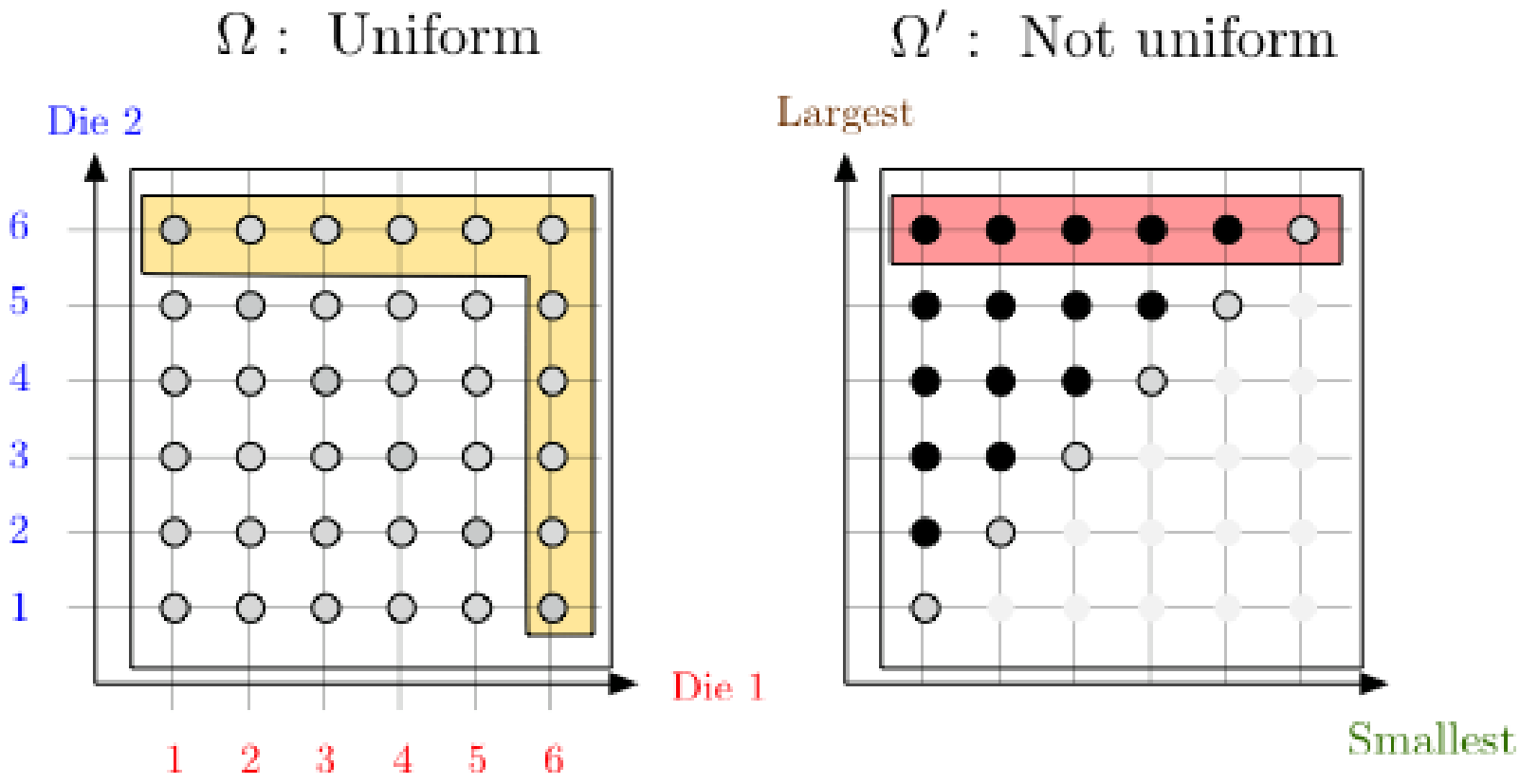

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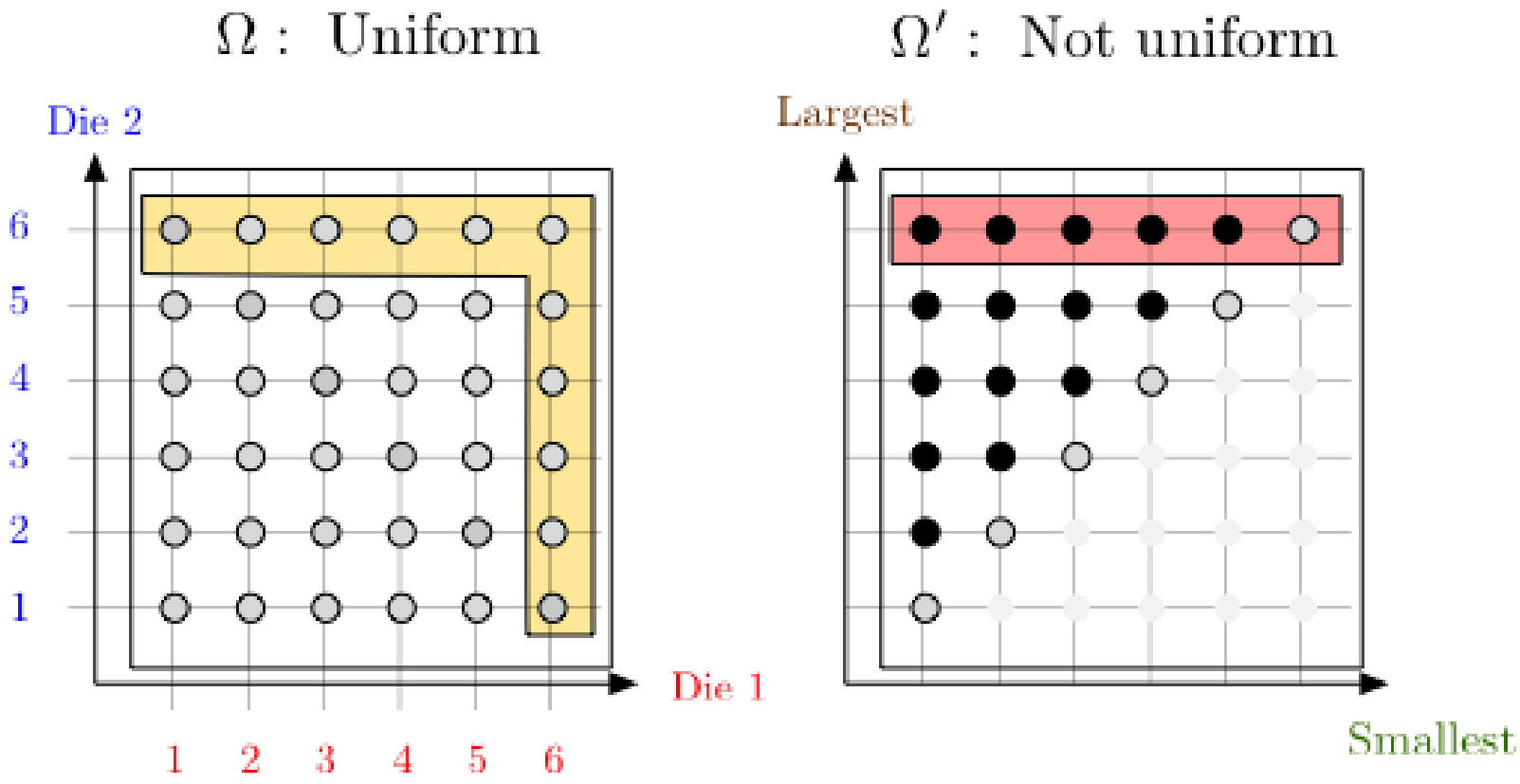

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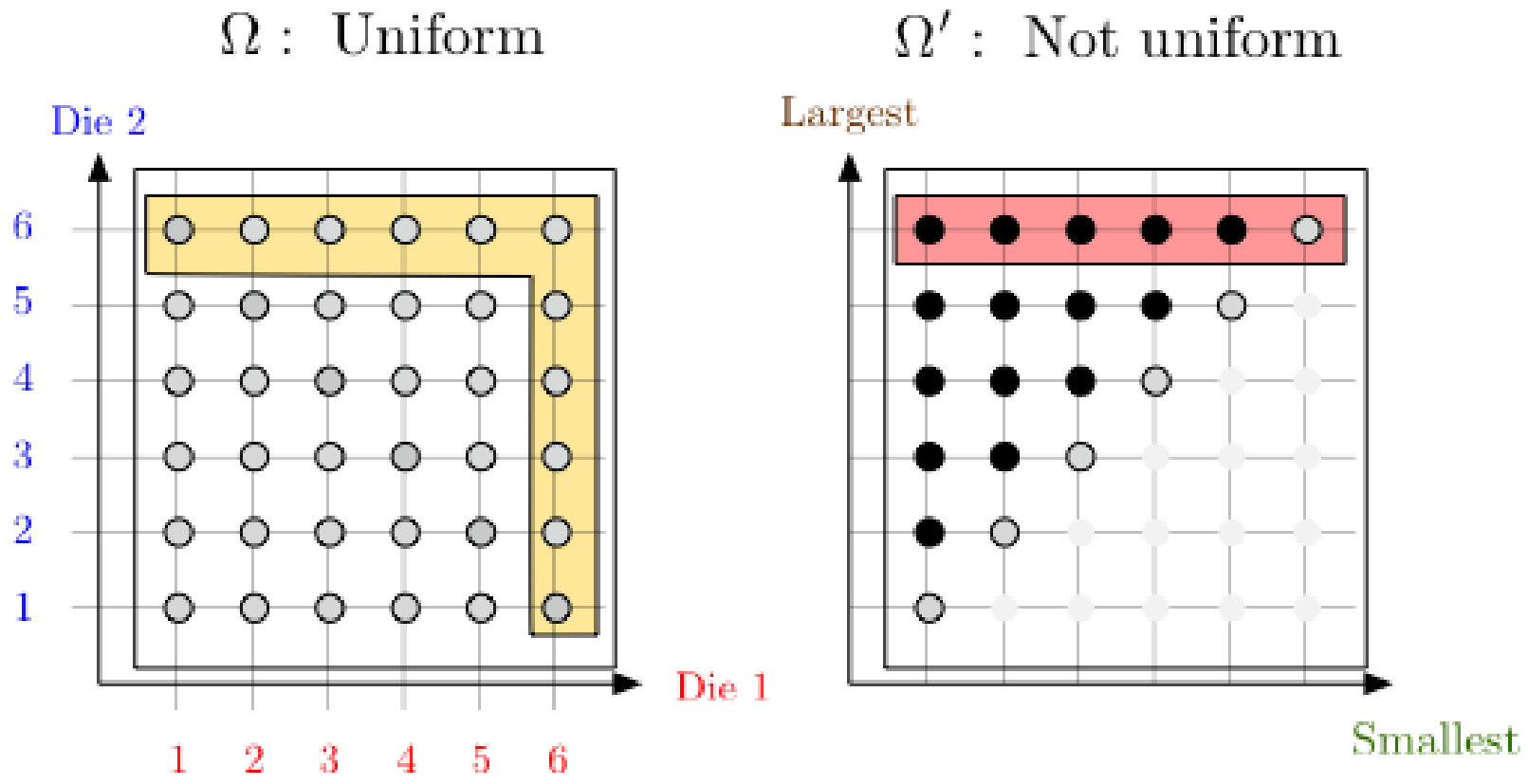

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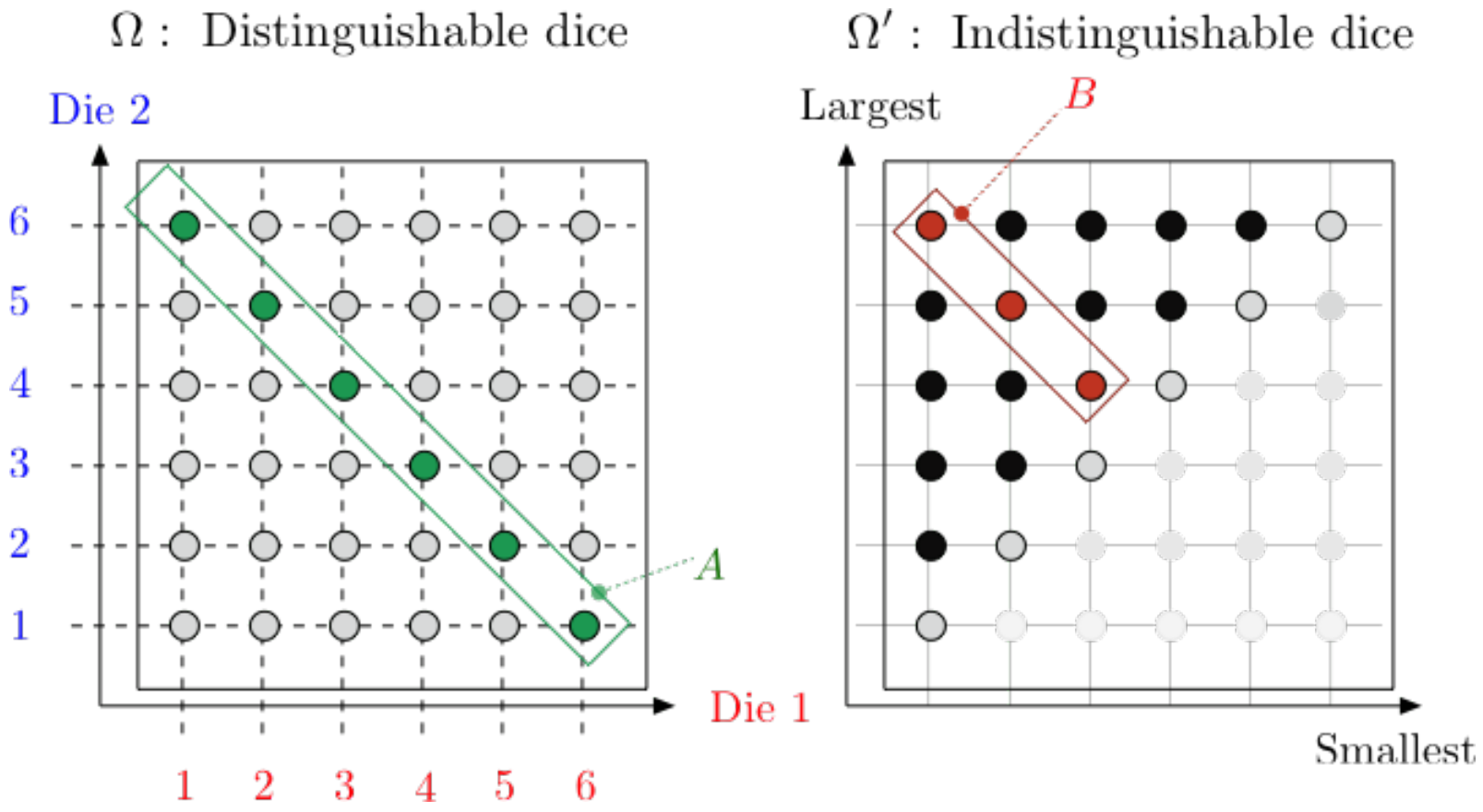

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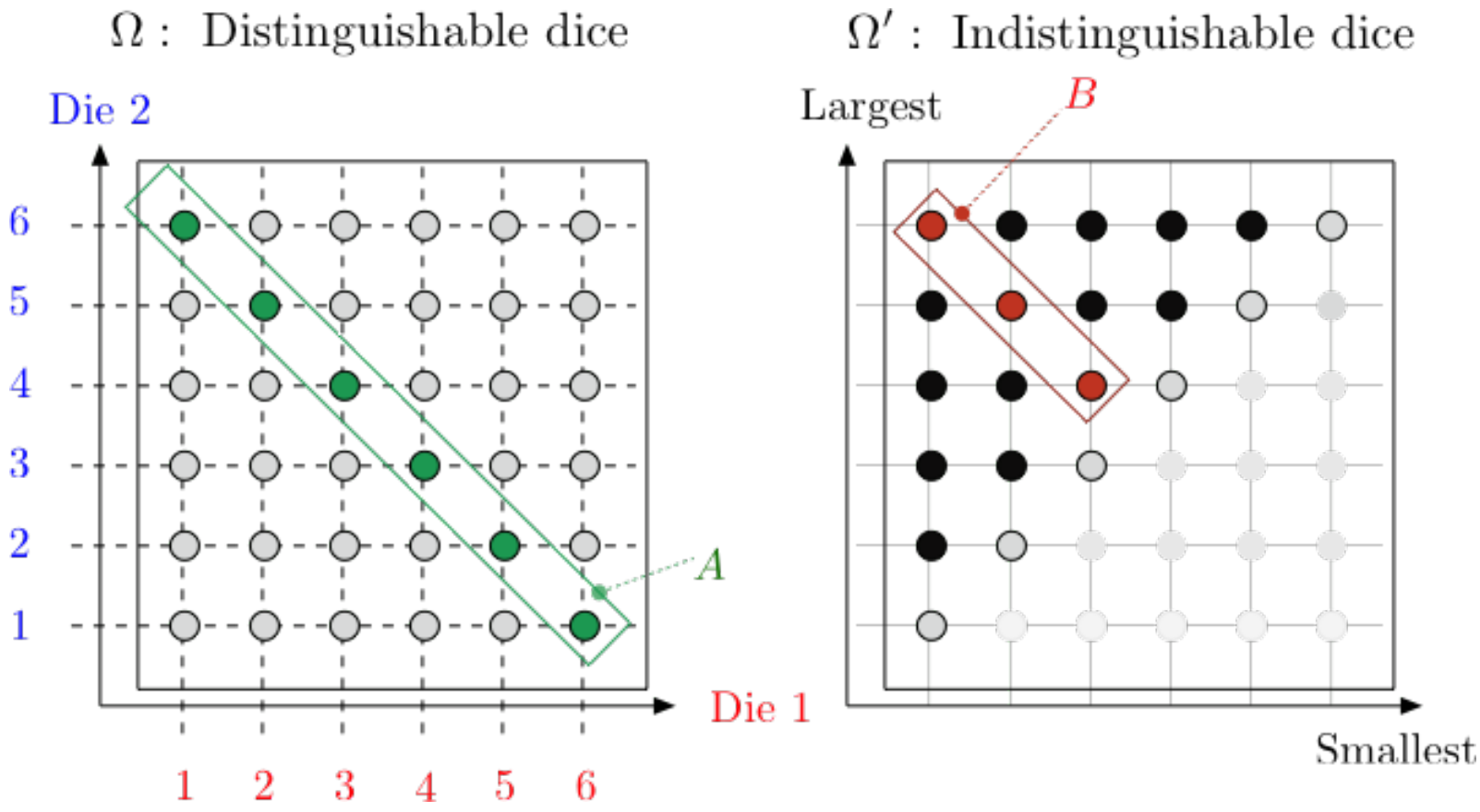

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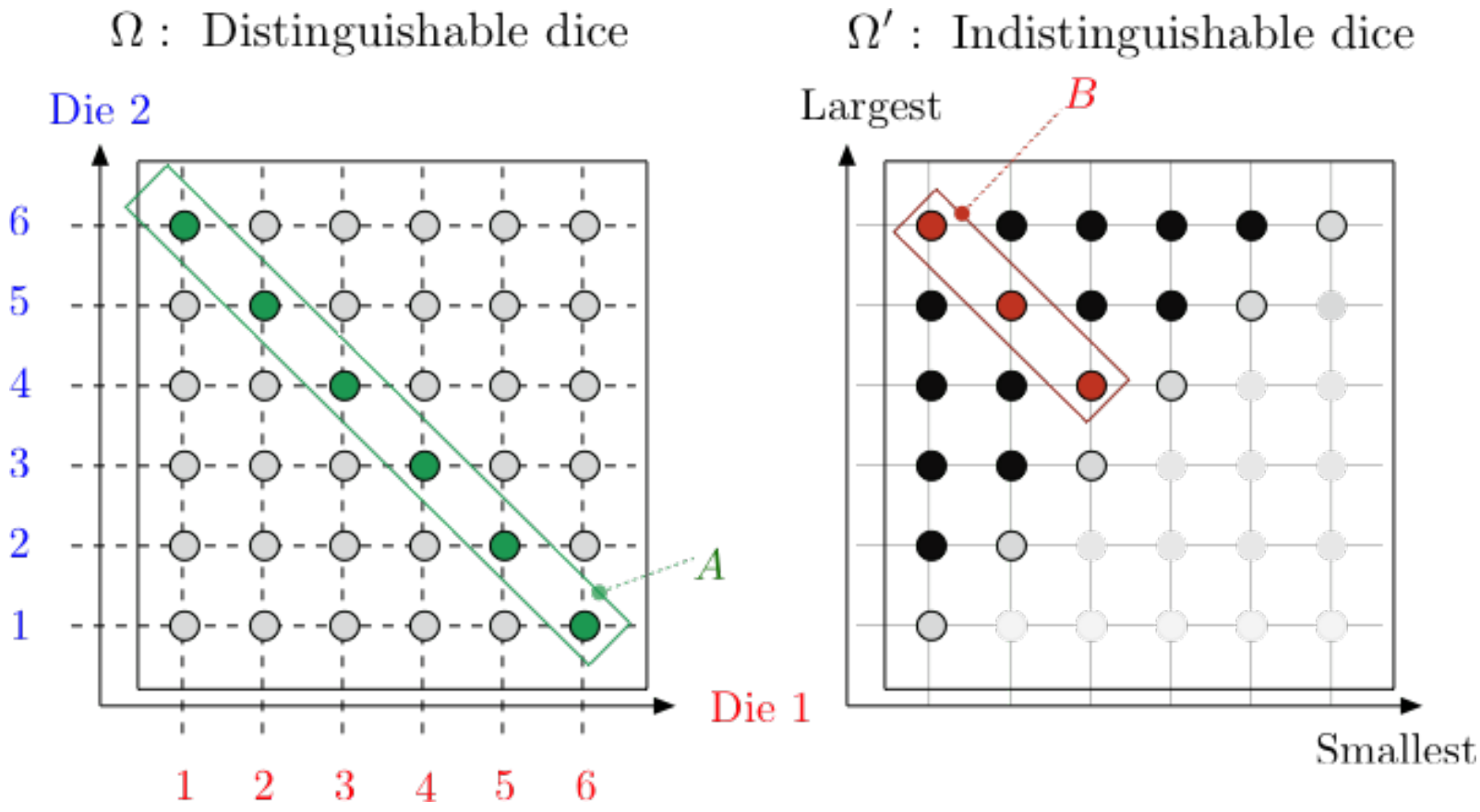

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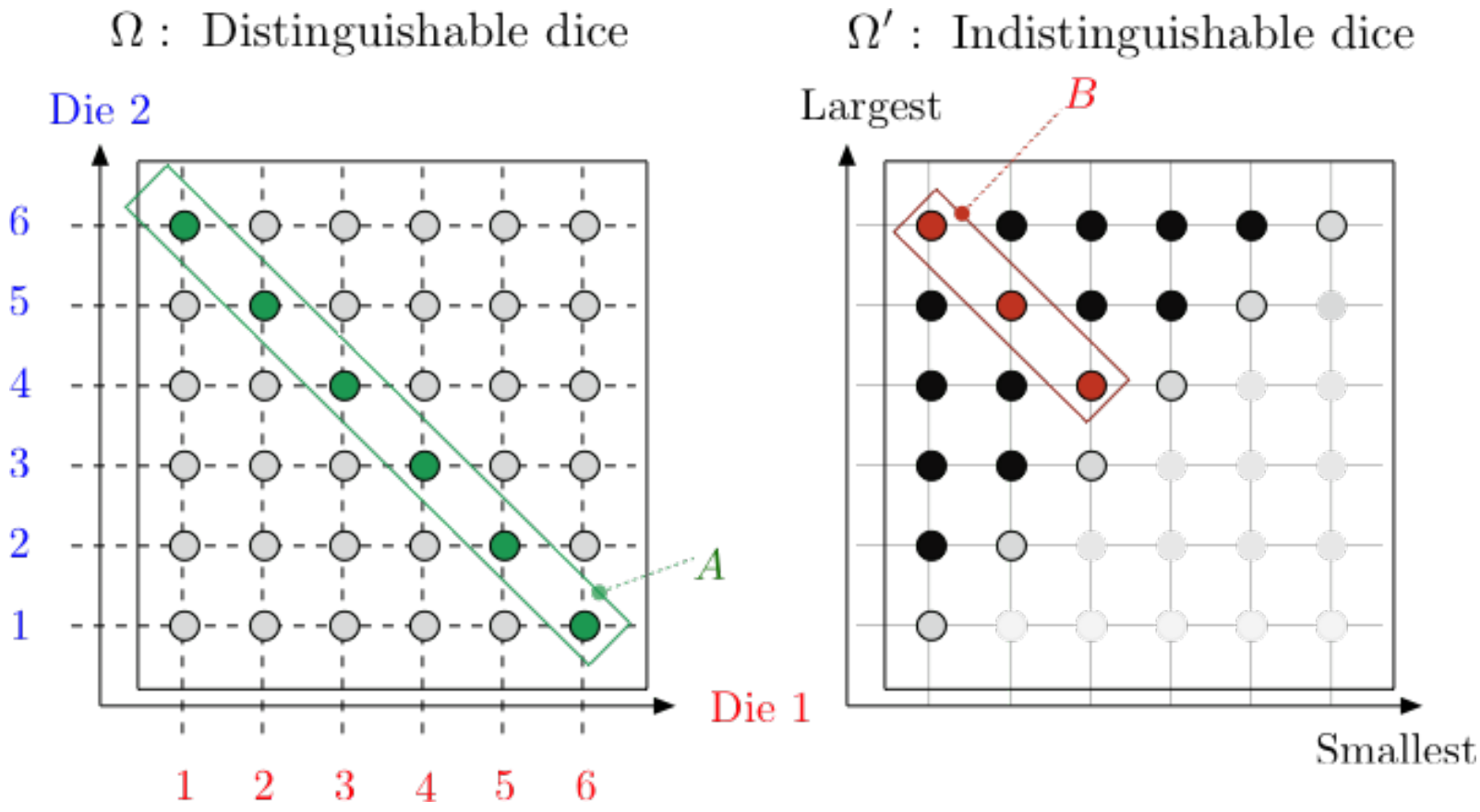

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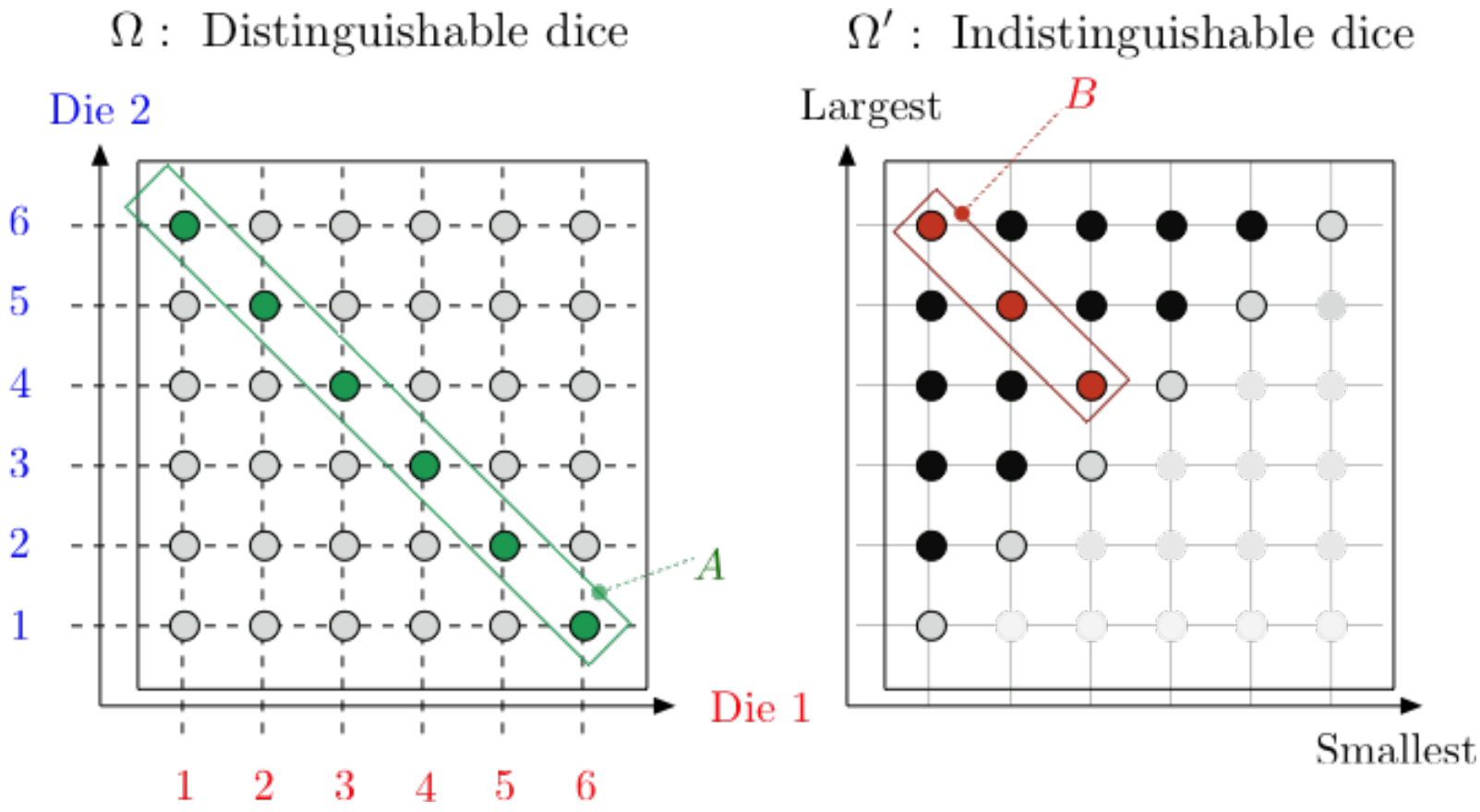

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  - Still sums to 1. Indeed $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$.

Set notation review

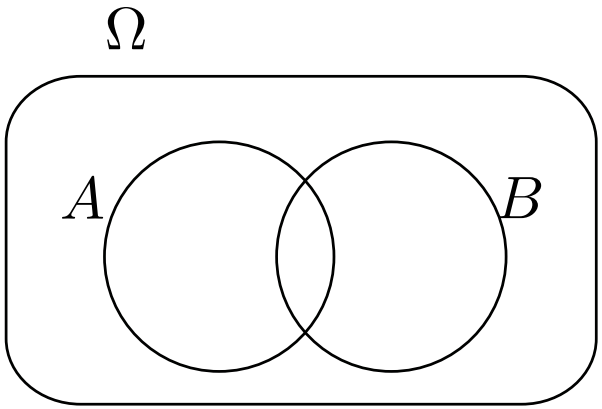

Figure : Two events

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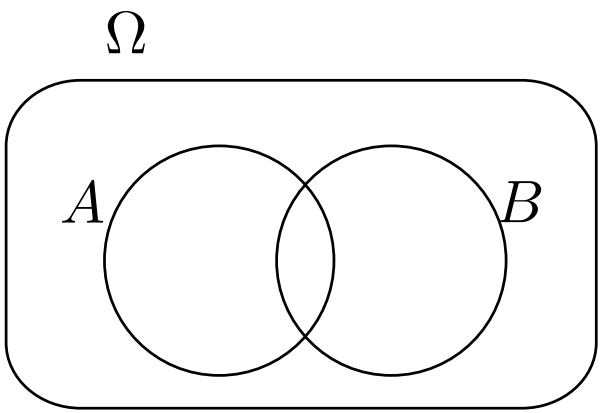

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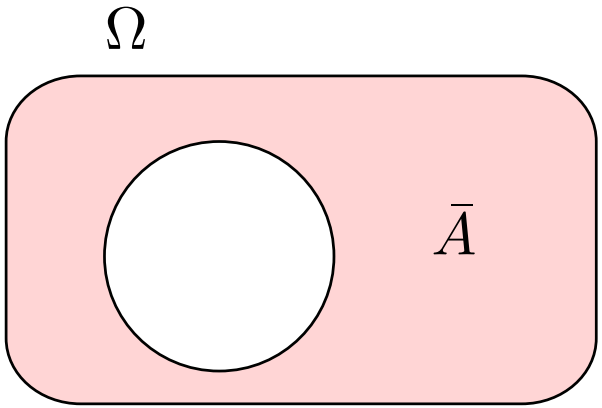

Figure : Complement (not)

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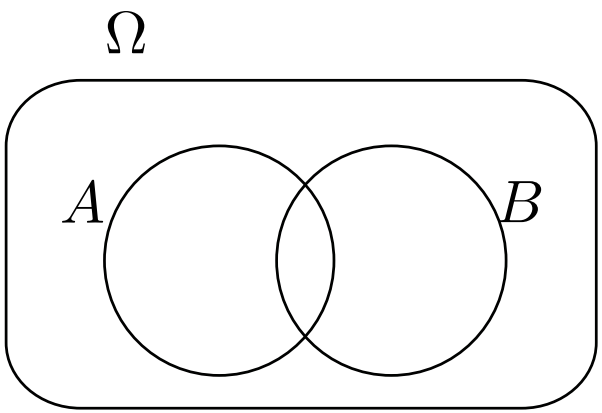

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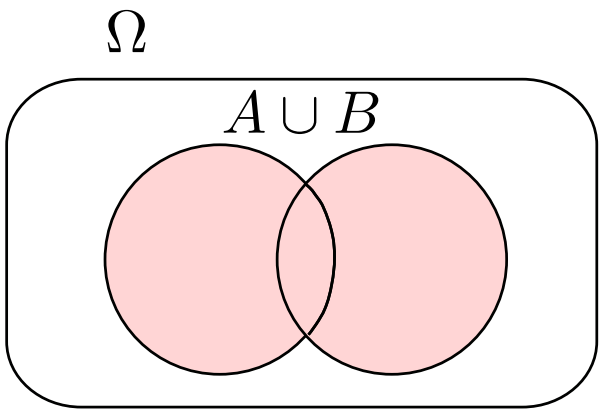
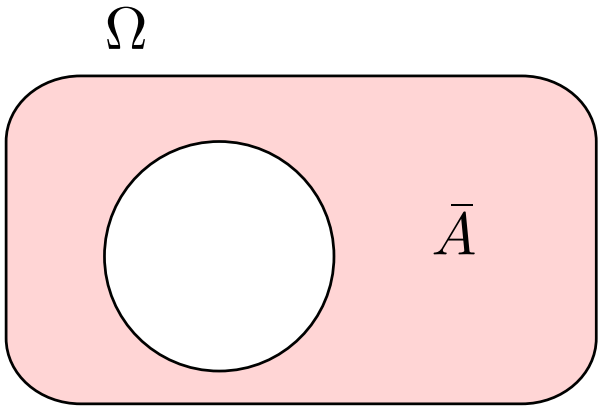


Figure : Union (or)



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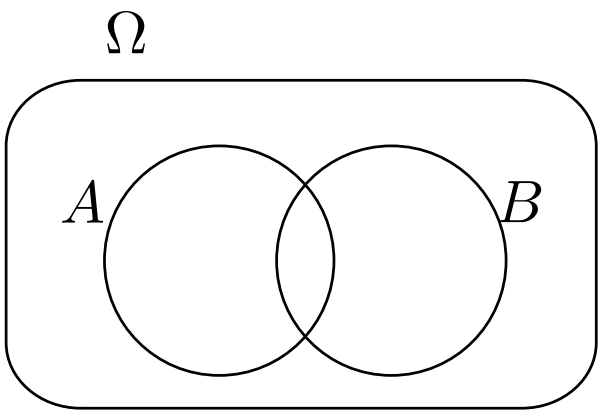

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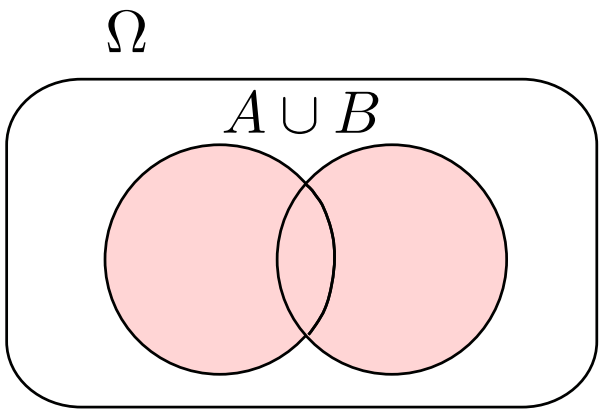

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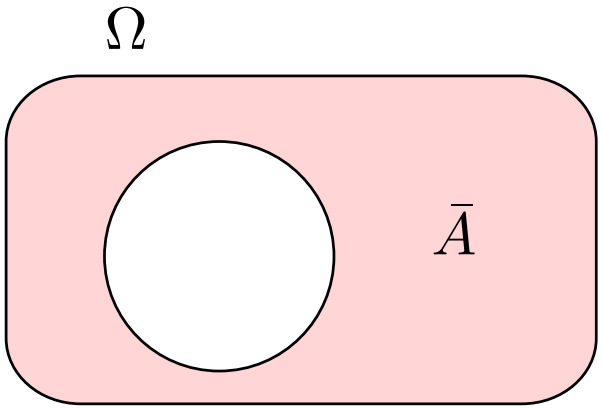

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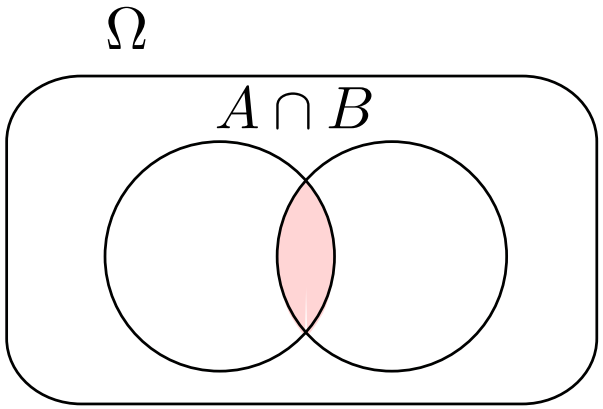

Figure : Intersection (and)

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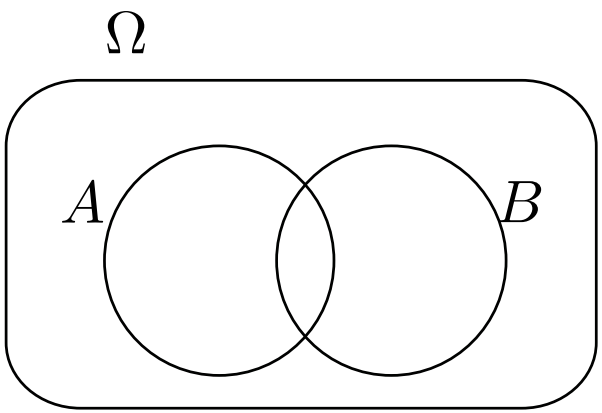

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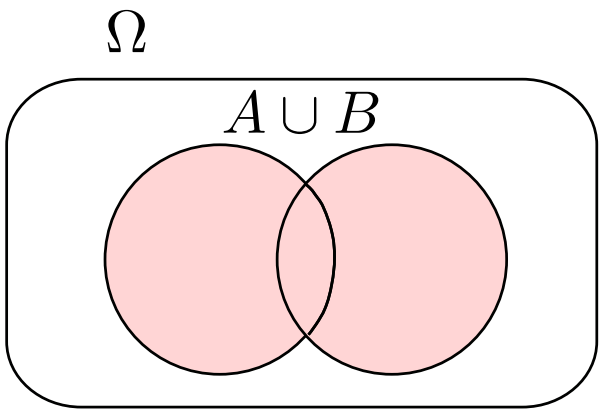

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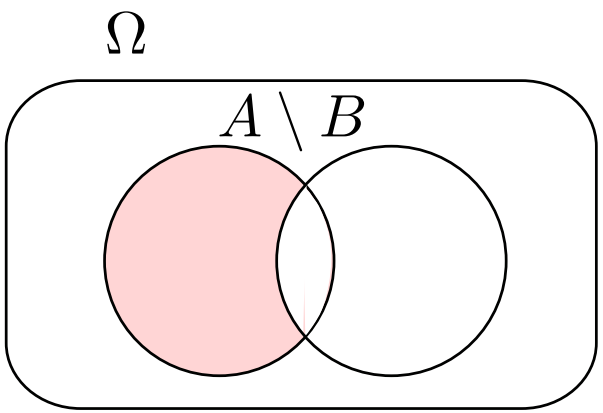
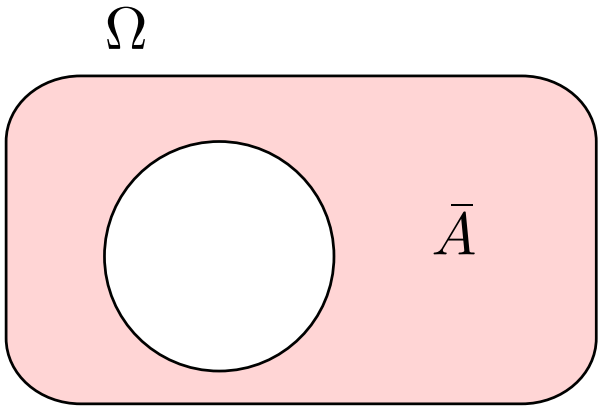


Figure : Difference (A, not B)



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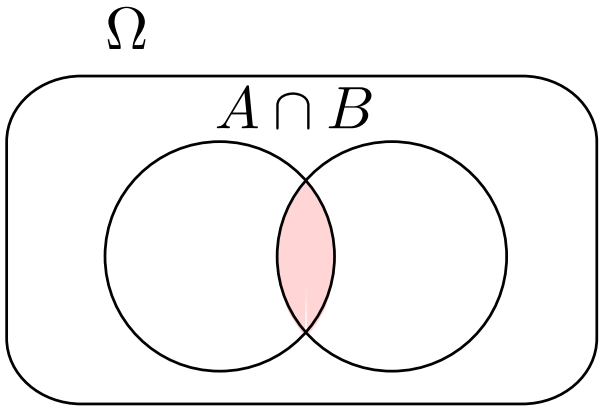

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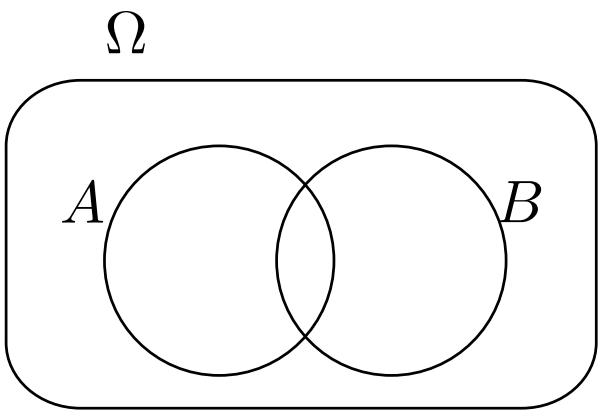

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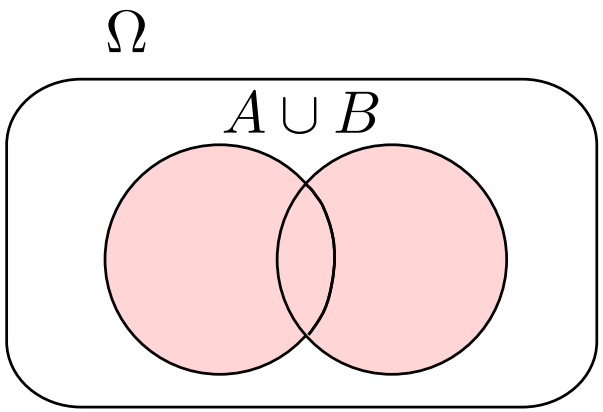

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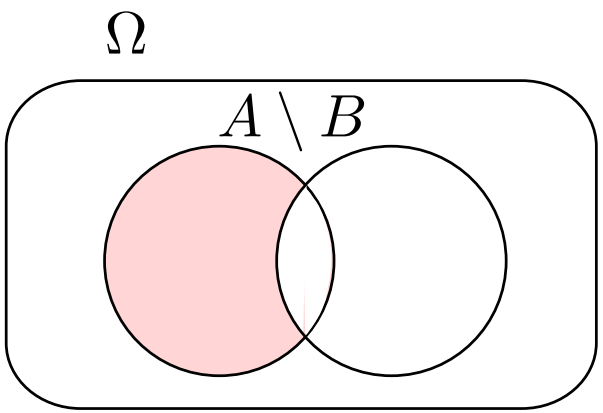

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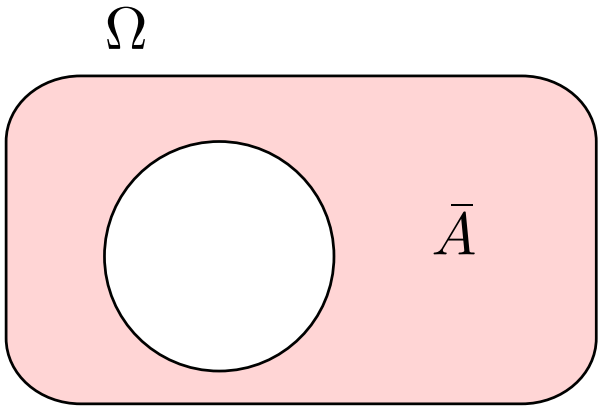

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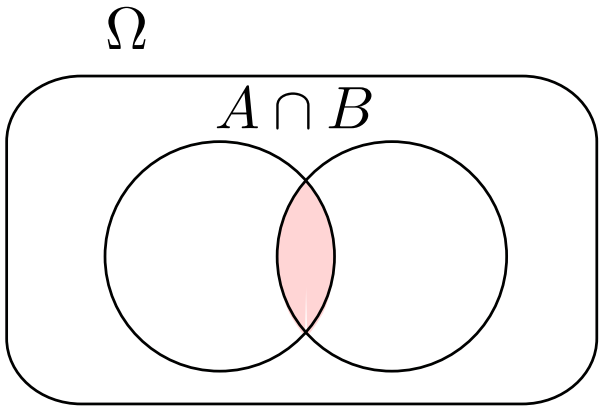

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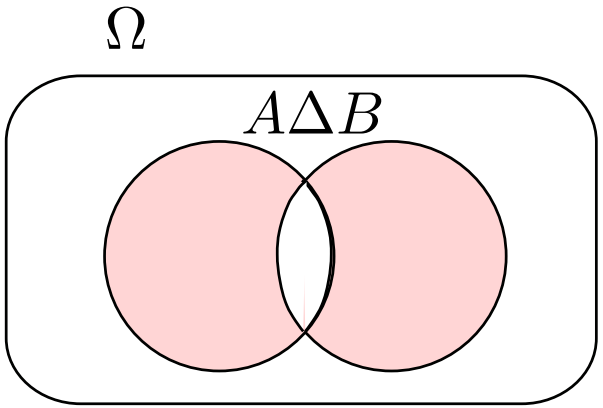

Figure : Symmetric difference (only one)

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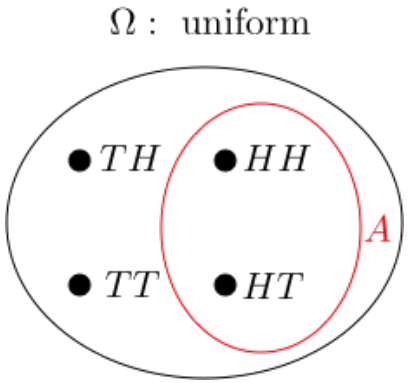

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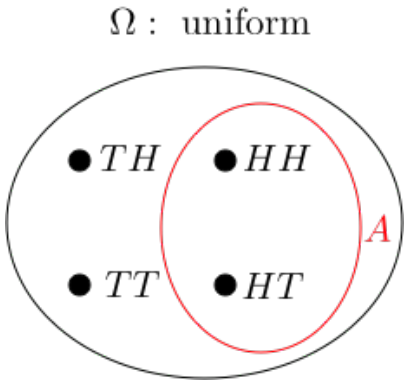

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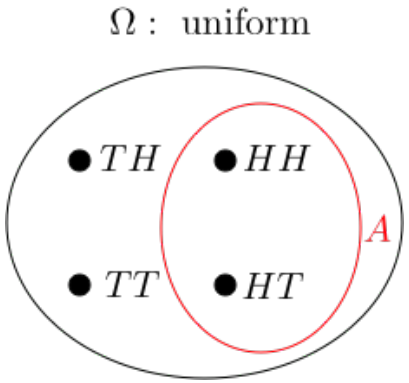

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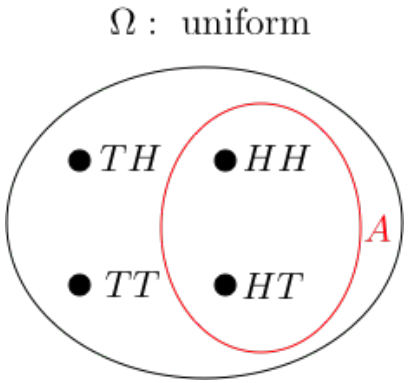

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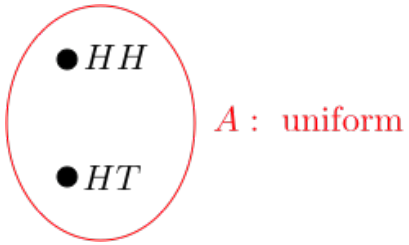

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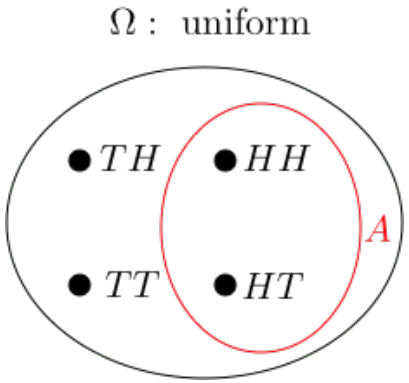

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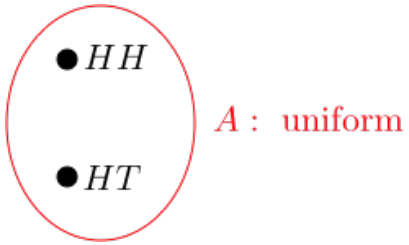

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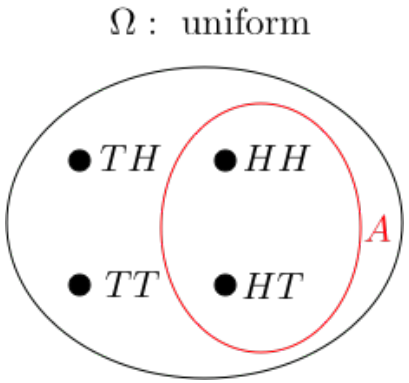

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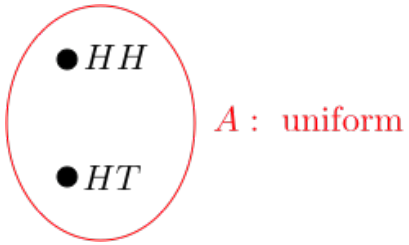

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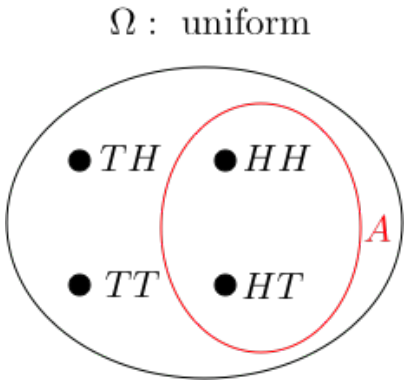

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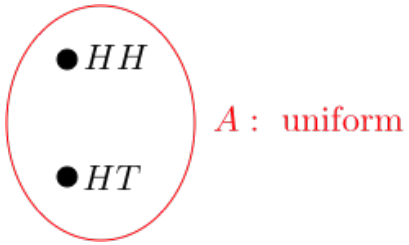

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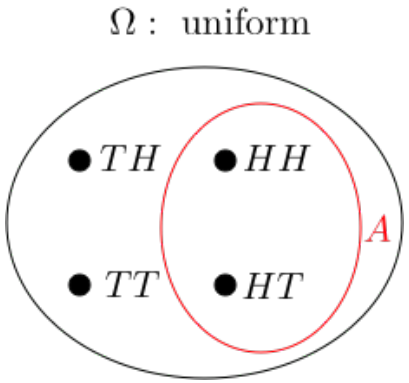

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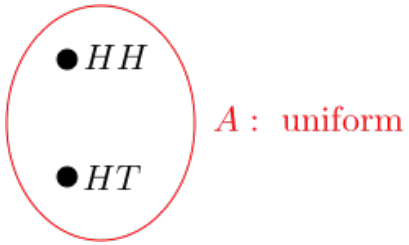

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The probability of two heads if the first flip is heads.
The probability of B **given** A is $1/2$.

A similar example.

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Two coin flips. One of the flips is heads.

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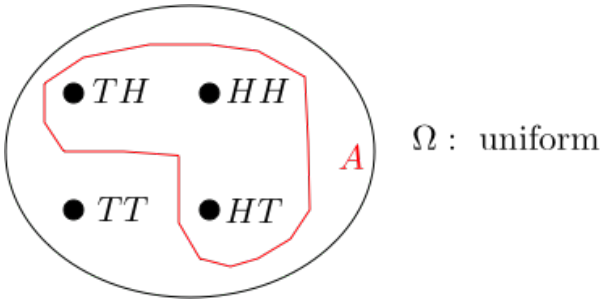

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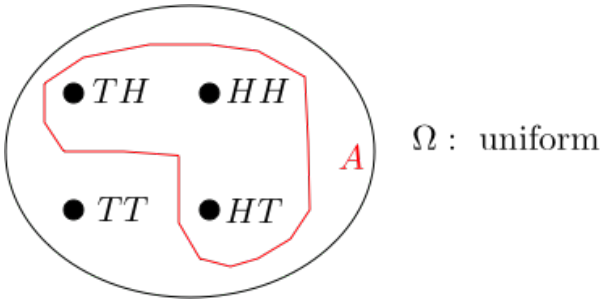

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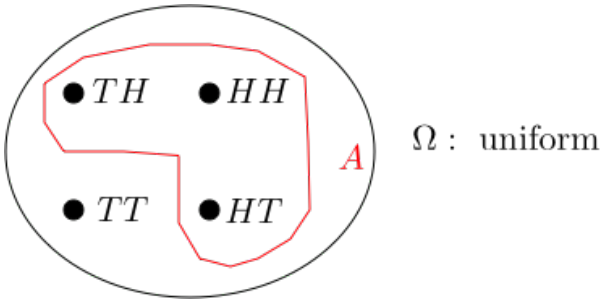

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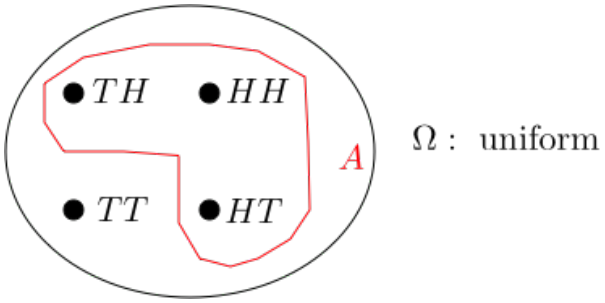

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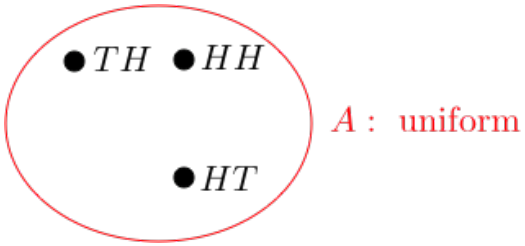

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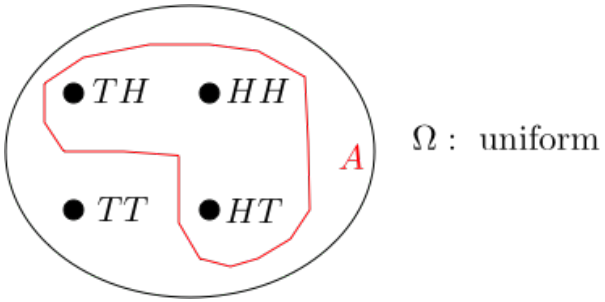

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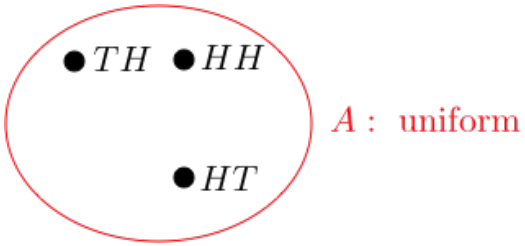

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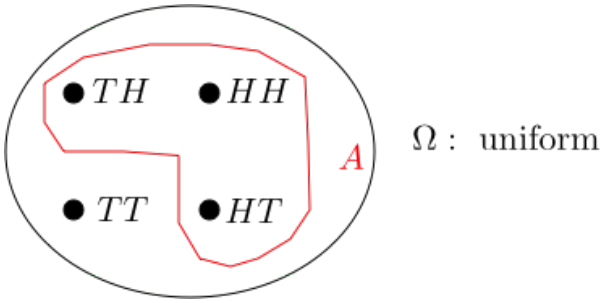

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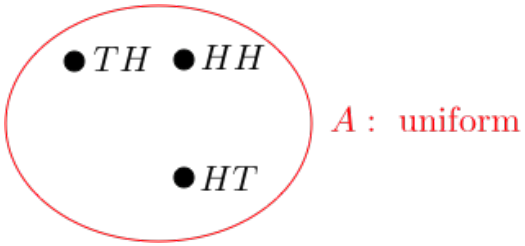

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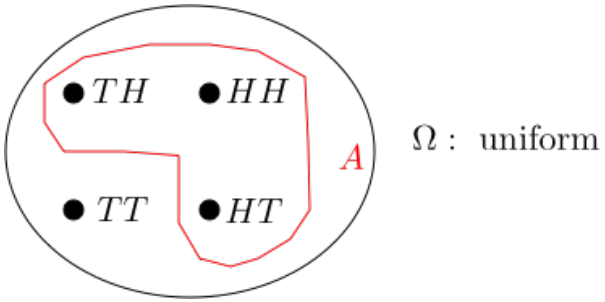

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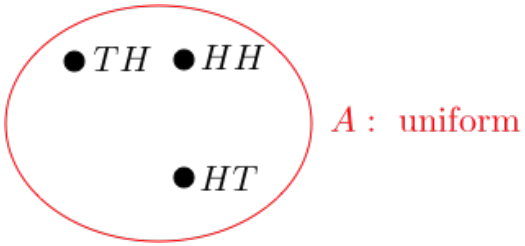

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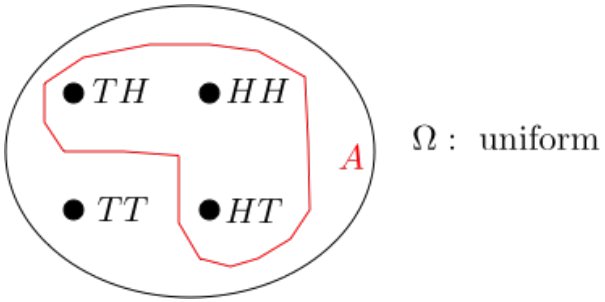

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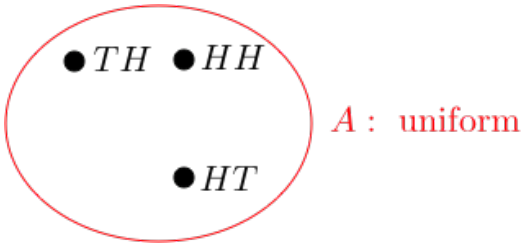

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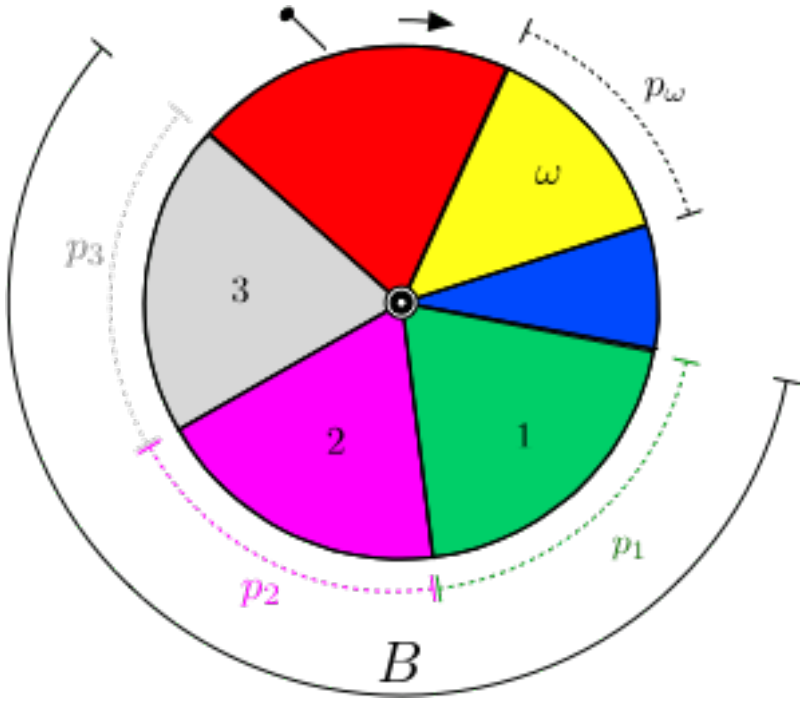

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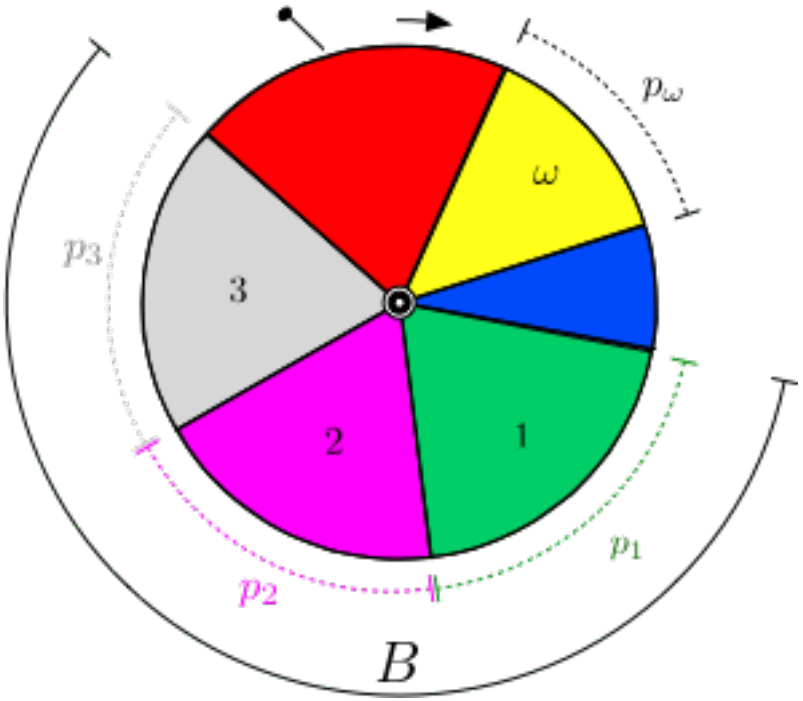

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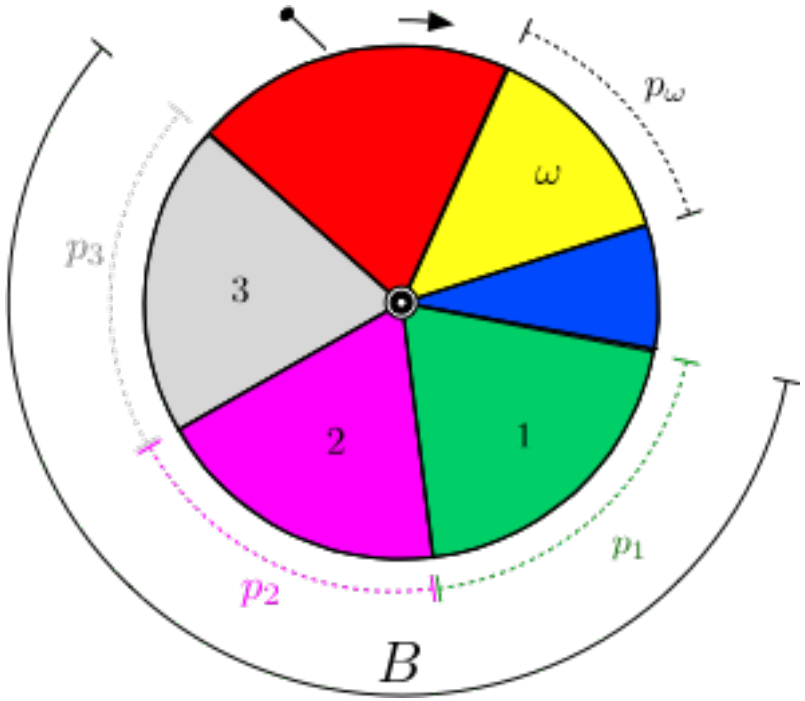

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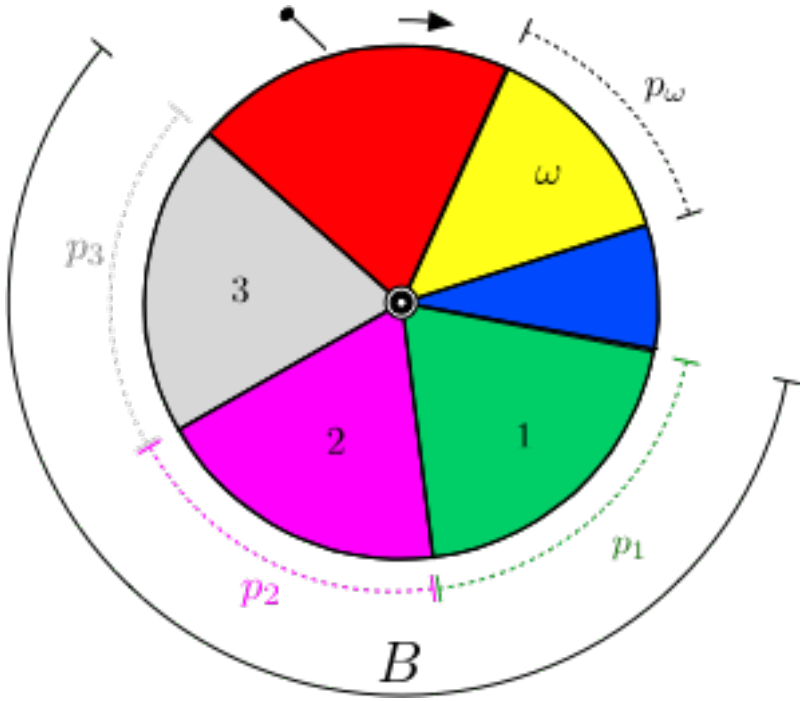

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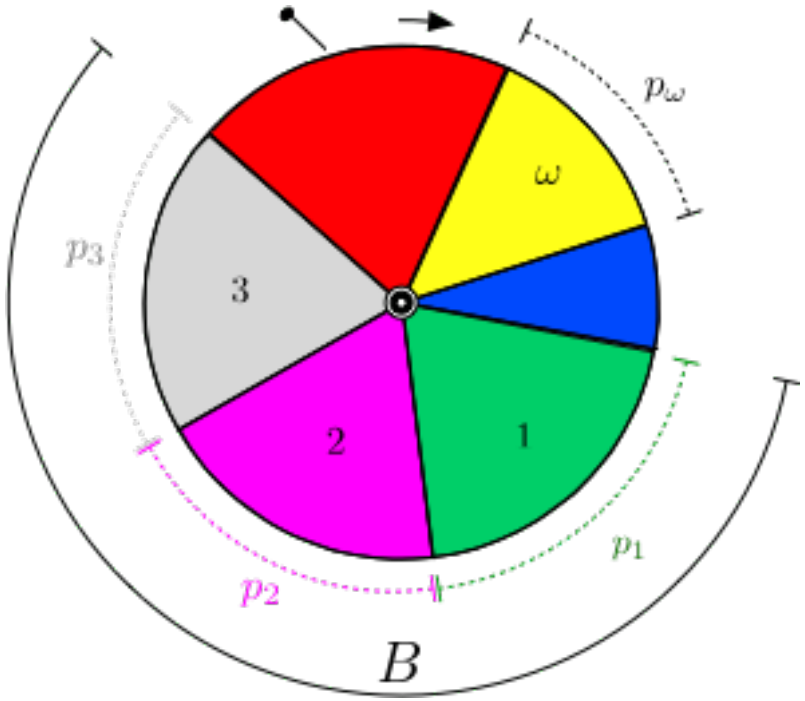

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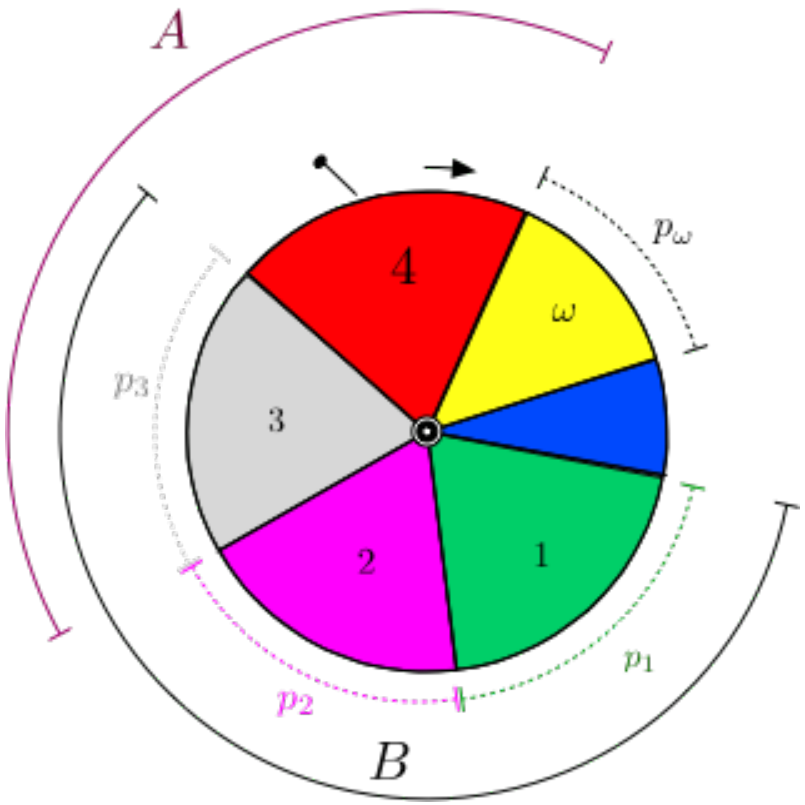

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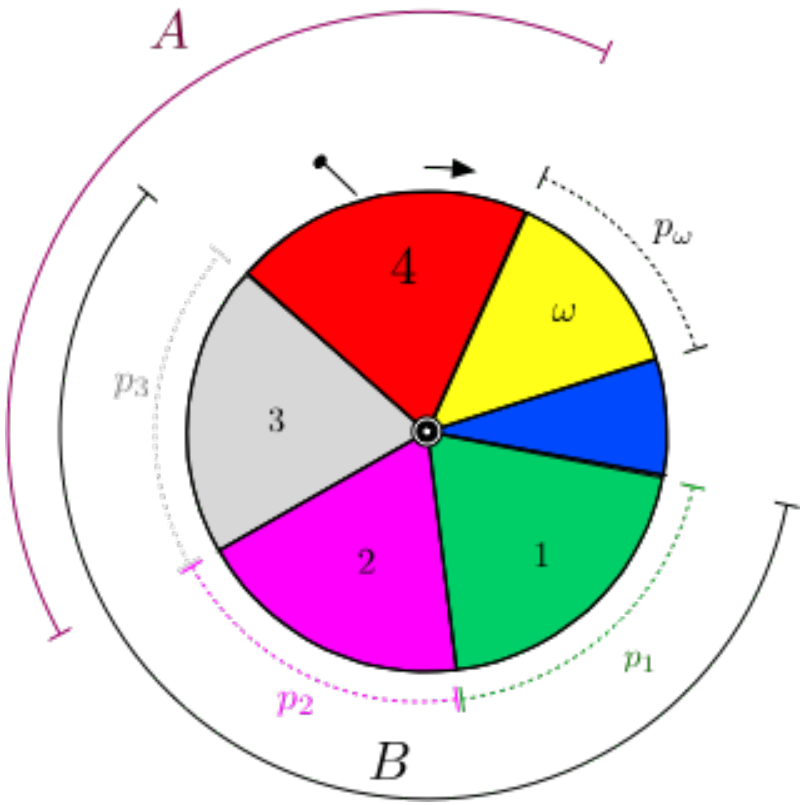

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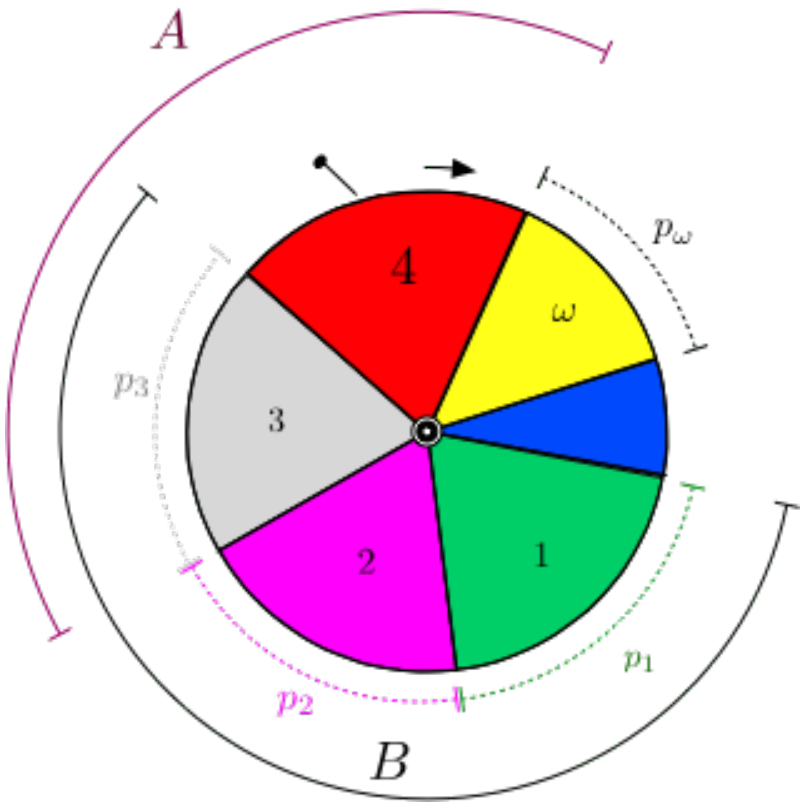

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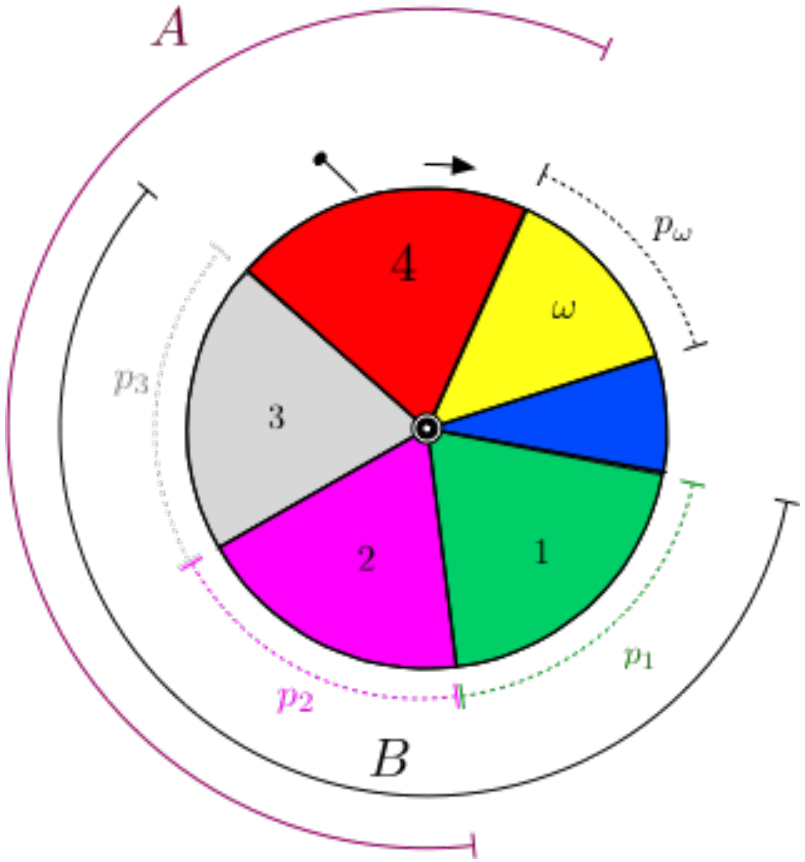

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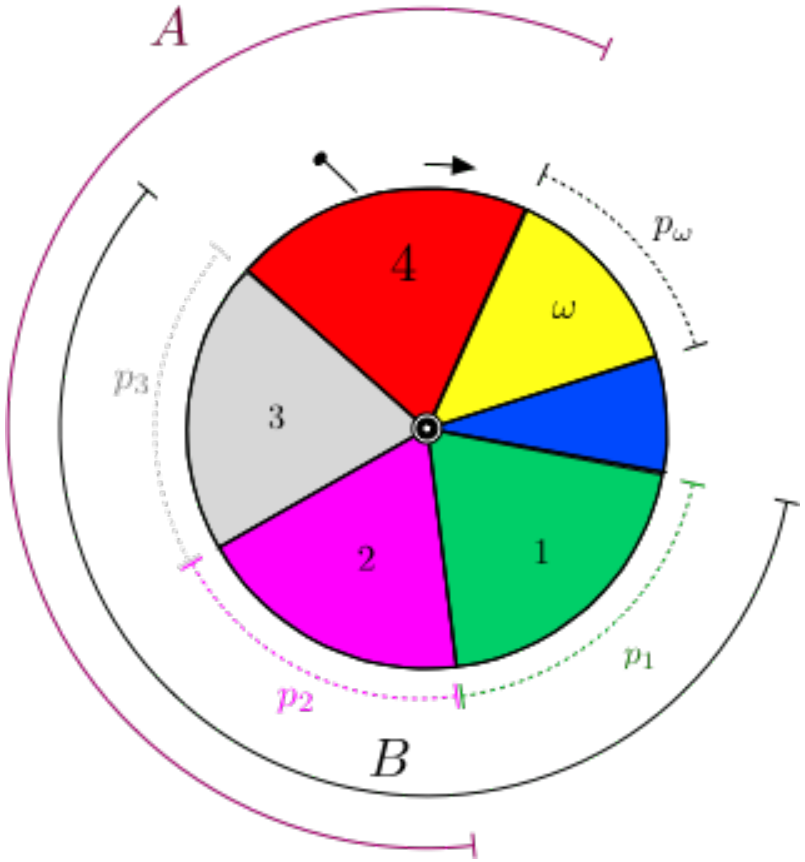

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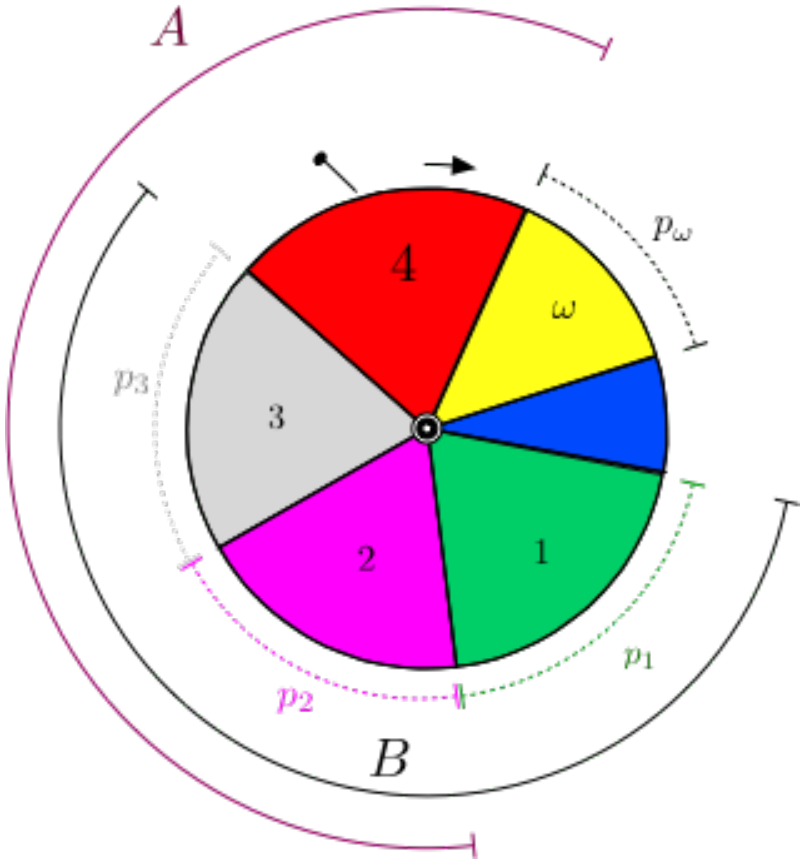

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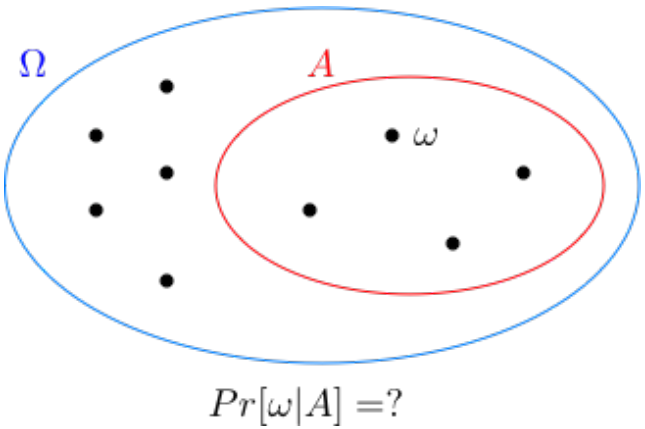

If in A, what is the probability of outcome ω?

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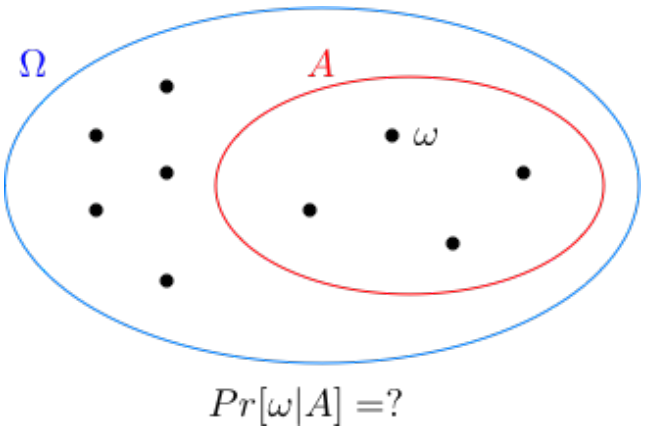

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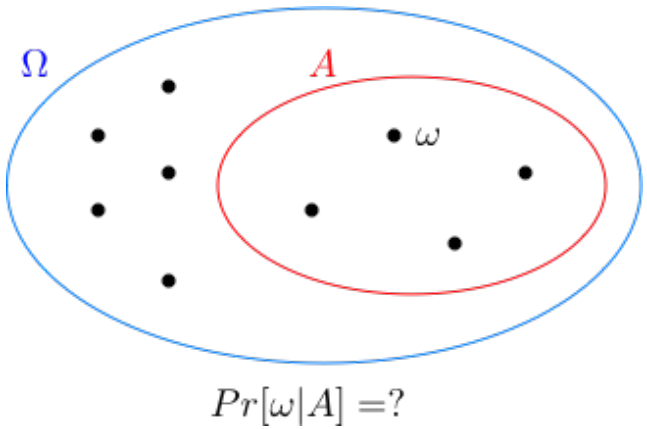

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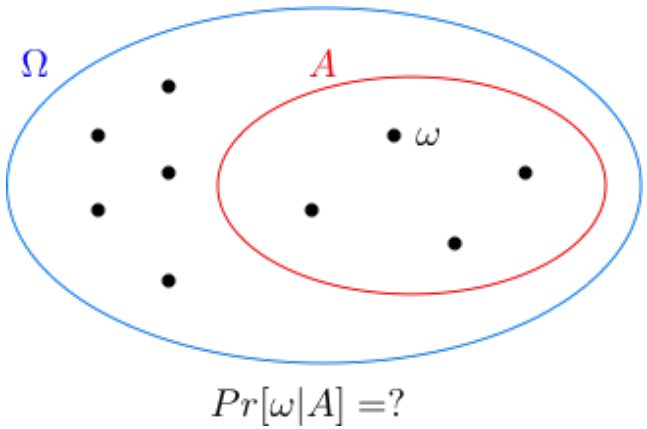

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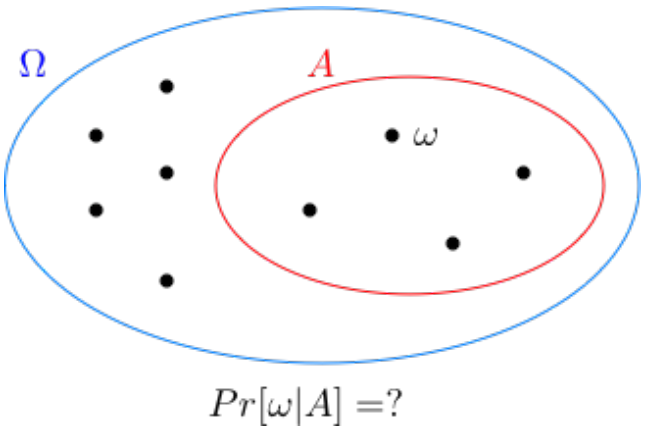

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Uniform Probability Space:

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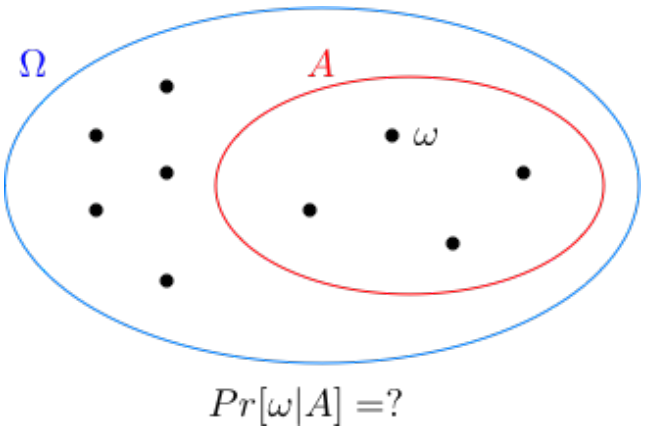

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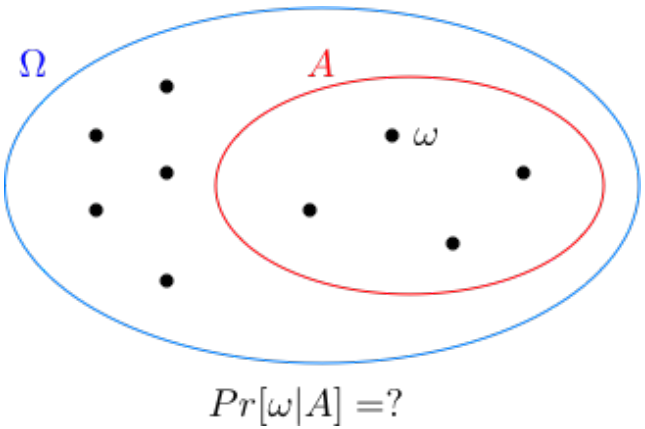

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Definition: The **conditional probability** of B given A is

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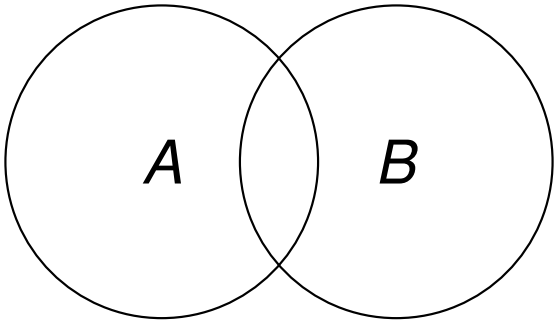

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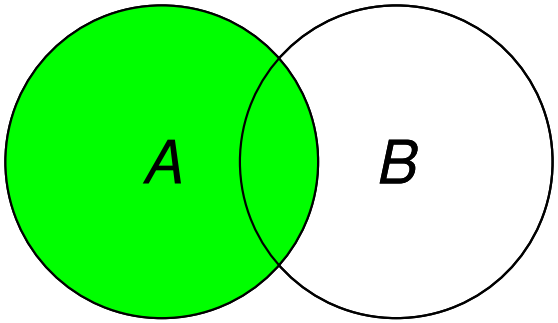

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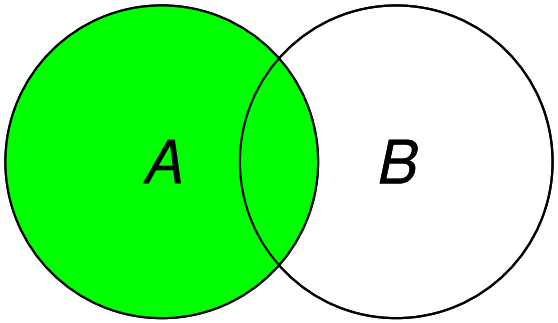

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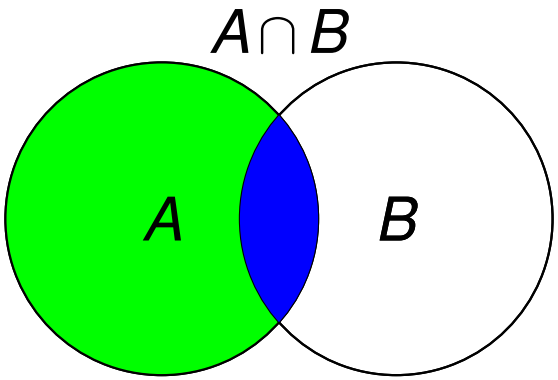

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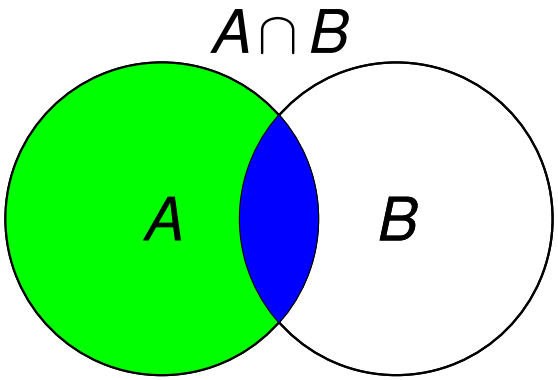

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