CS70: Jean Walrand: Lecture 27.

Expectation; Conditional Expectation; B(n, p); G(p)

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1. Review of Expectation
2. Linearity of Expectation
3. Conditional Expectation
4. Independence of RVs
5. Applications
6. Important Distributions and Expectations.

Expectation

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The random variable X is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

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$\square$

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Hence,

$$E[X] = \frac{7n}{2}.$$

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Note that linearity holds even though the X_m are not independent (whatever that means).

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Flip n coins with heads probability p.

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Moreover $X = X_1 + \cdots X_n$ and

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Flip n coins with heads probability p. X - number of heads

Binomial Distibution: $Pr[X = i]$, for each i.

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}.$$

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Uh oh. ... Or... a better approach: Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

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Conditional Expectation

How do observations affect expectation?

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Hence, the mean value is 4.5.

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Is this a coincidence?

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Recall the distribution of X: $Pr[X = 2] = Pr[X = 12] = 1/36, Pr[X = 3] = Pr[X = 11] = 2/36, \ldots$.

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$$\{(8, 5/15), (9, 4/15), (10, 3/15), (11, 2/15), (12, 1/15)\}.$$

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Hence,

$$E[X|X \geq 8] = 8\frac{5}{15} + 9\frac{4}{15} + 10\frac{3}{15} + 11\frac{2}{15} + 12\frac{1}{15} = \frac{140}{15} \approx 9.33.$$

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Roll two dice. You are told that the total number X of pips is less than 8. What is the expected value of X given that information?

We find that

$$E[X|X<8] = 2\frac{1}{21} + 3\frac{3}{21} + \cdots + 7\frac{6}{21} = \frac{112}{21} \approx 5.33.$$

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Coincidence?

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Coincidence? Probably not.

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Note that $\Omega = \{\omega_n, n = 1, 2, \ldots\}$.

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Let X be the number of flips until the first H. Then, $X(\omega_n) =$

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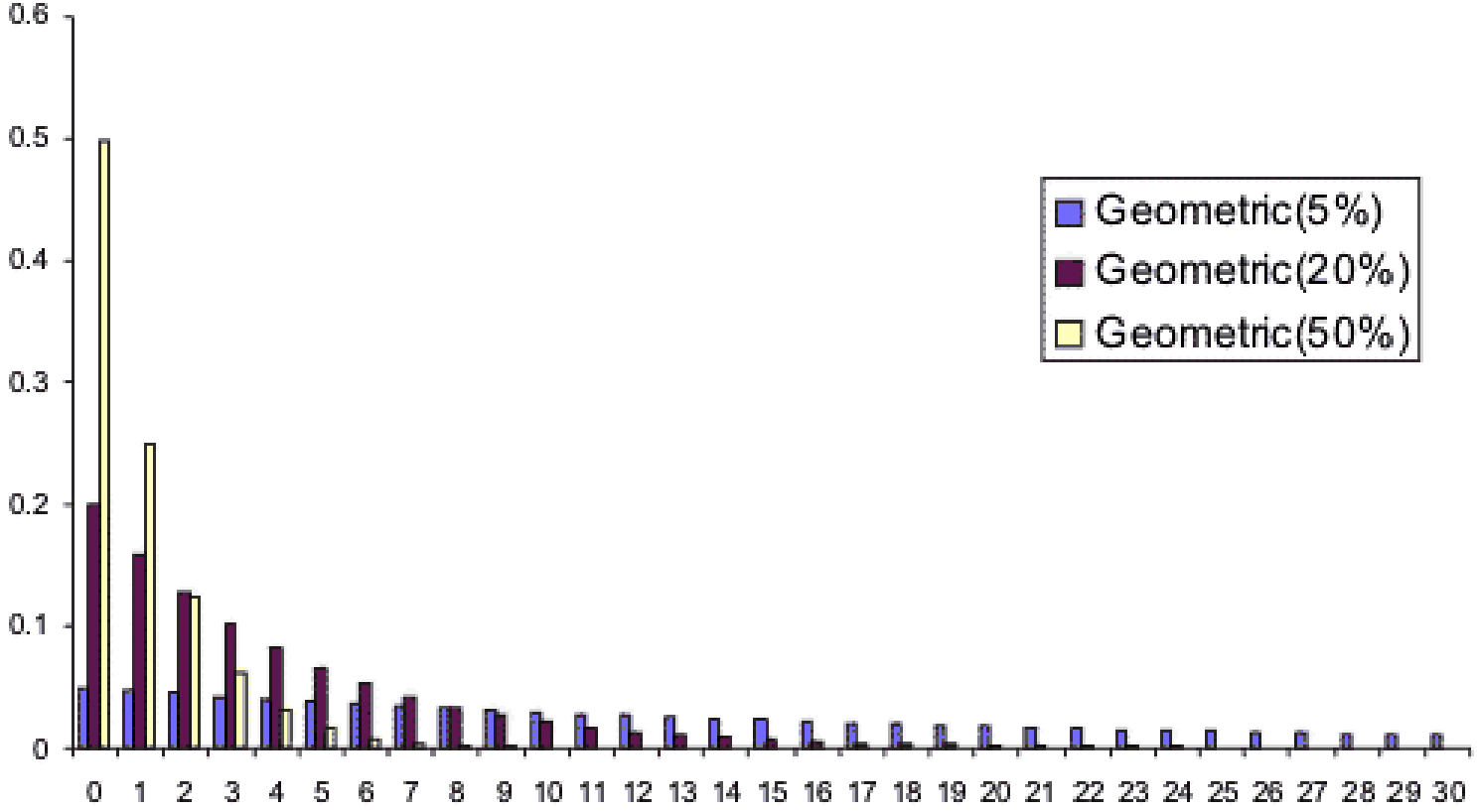

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A different look at the algebra.

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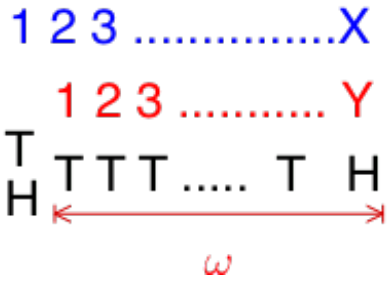

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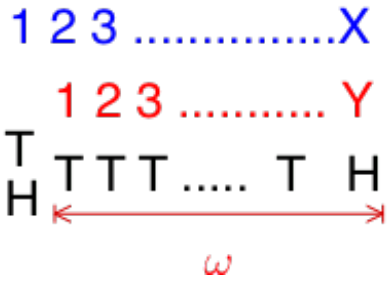

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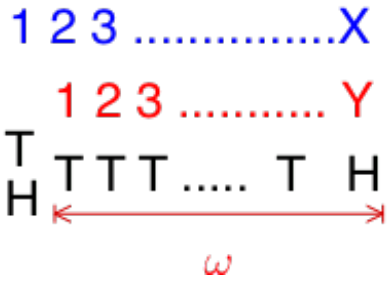

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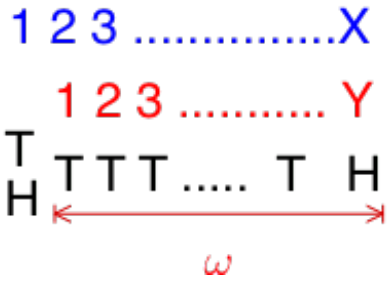

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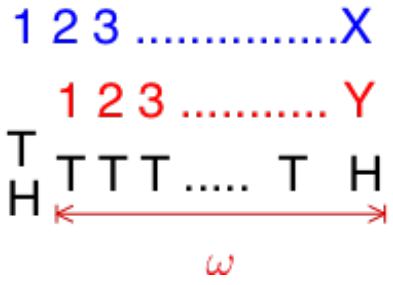

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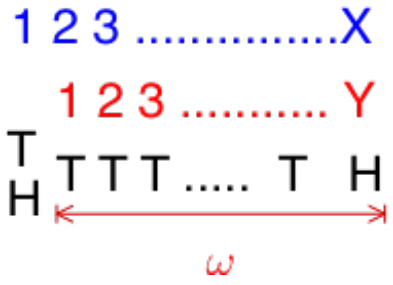

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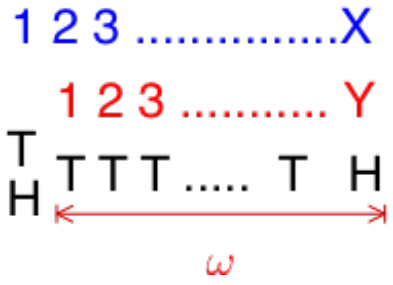

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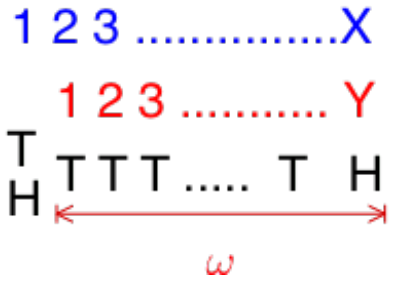

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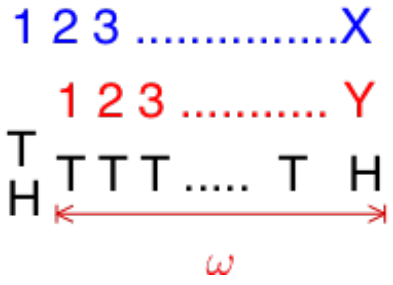

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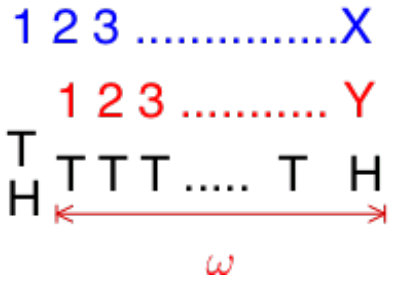

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Let X be $G(p)$. Then, for $n \geq 0$,

$$Pr[X > n] = Pr[\text{ first } n \text{ flips are } T] = (1-p)^n.$$

Theorem

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

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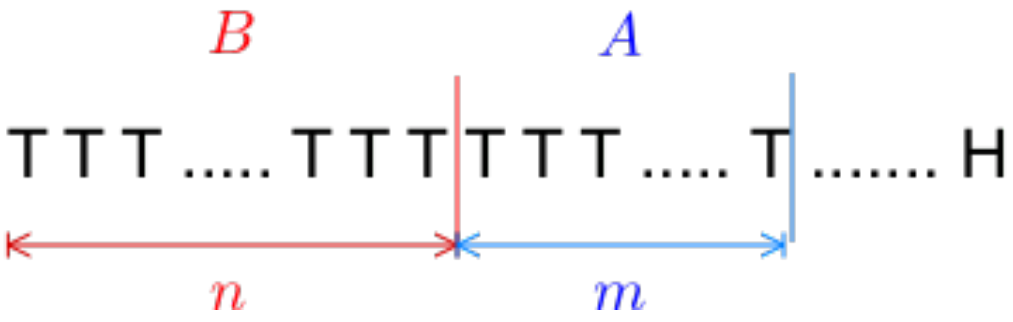

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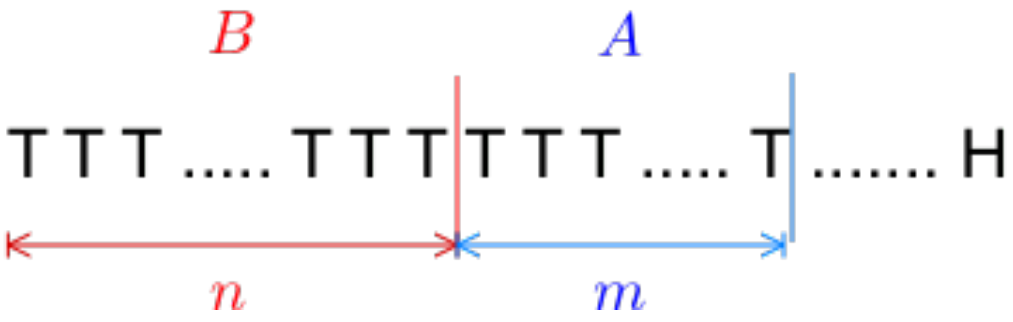

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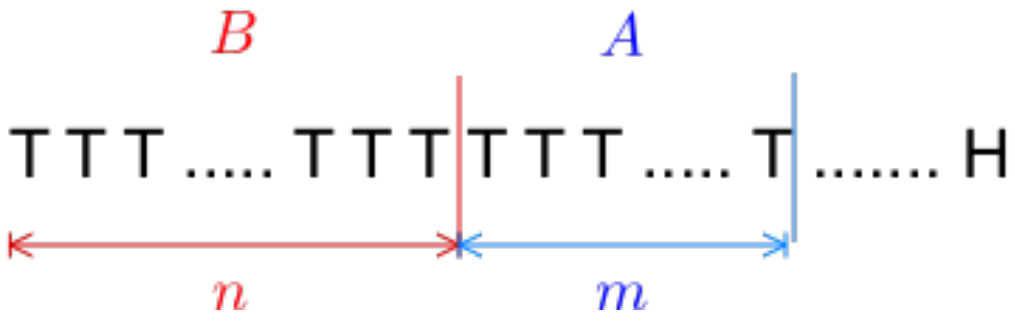

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The coin is memoryless, therefore, so is X.

Geometric Distribution: Yet another look

Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

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$\square$

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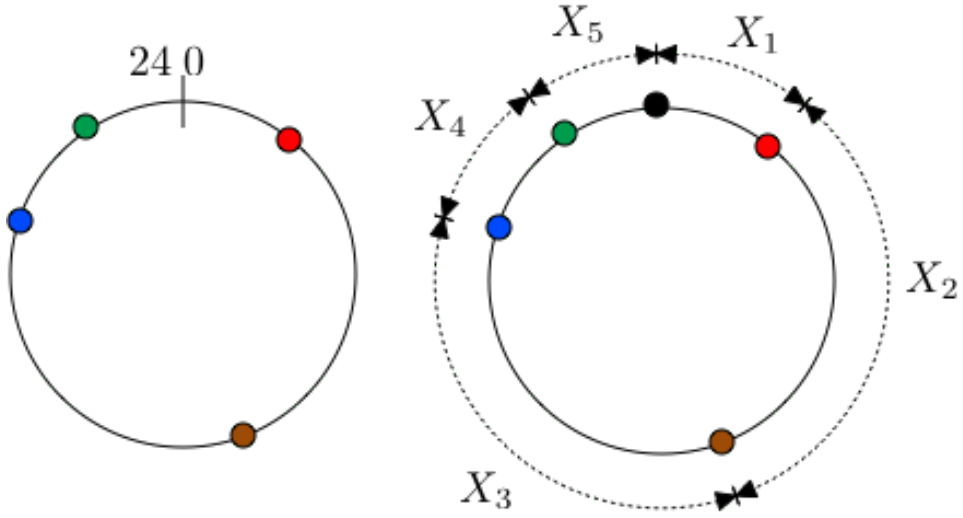
Here is an alternative picture (left)



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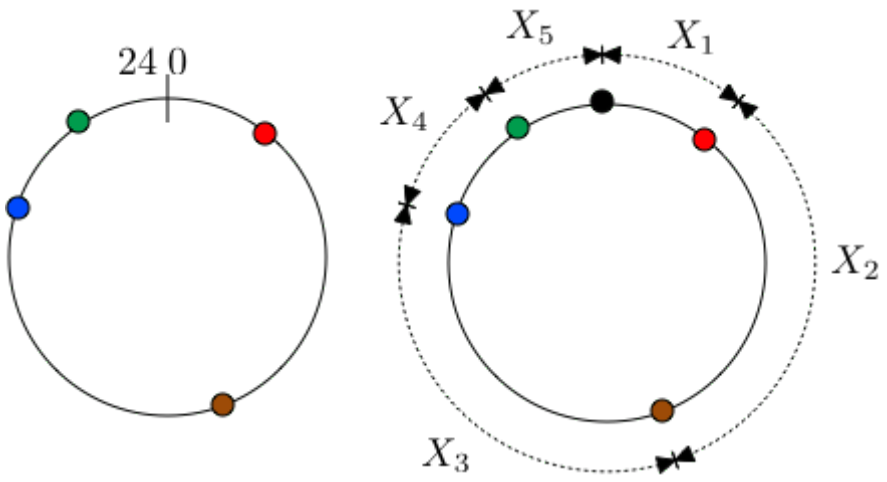

Add the black dot uniformly at random and pretend that it represents 0/24.

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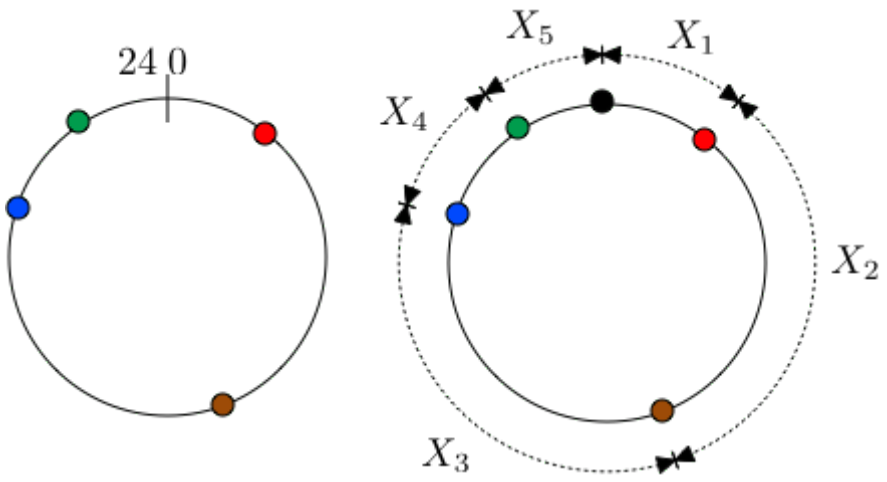

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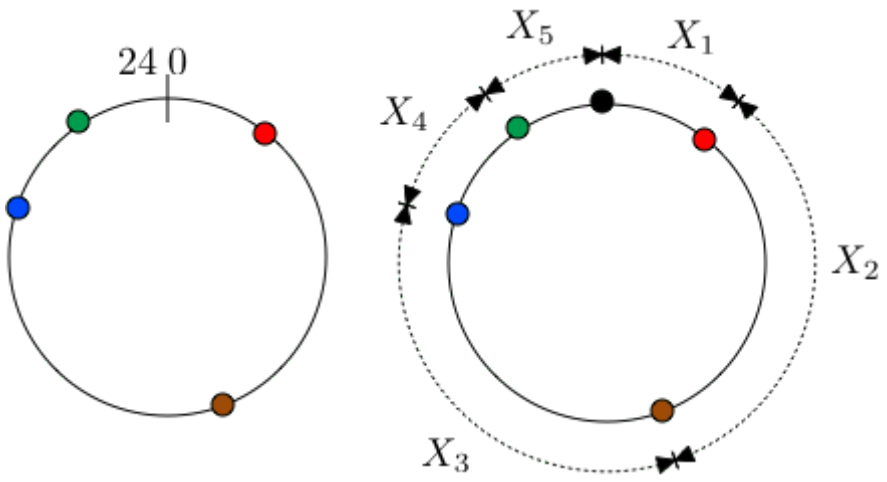

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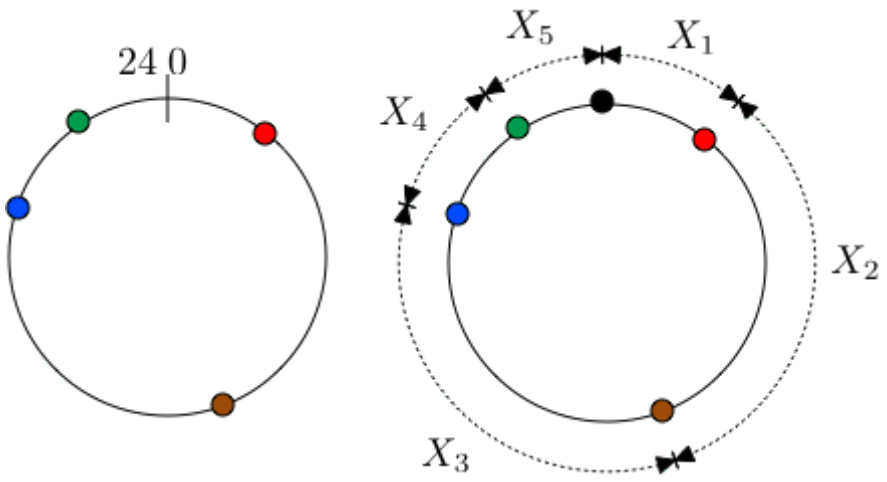

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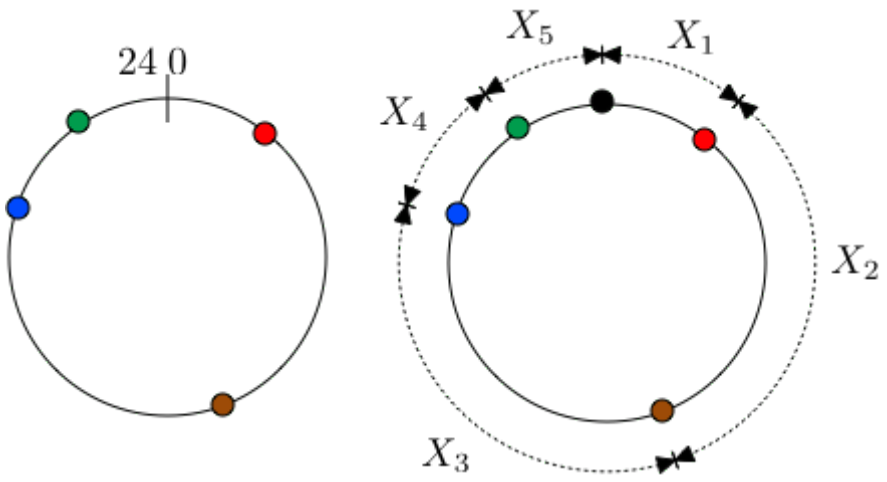

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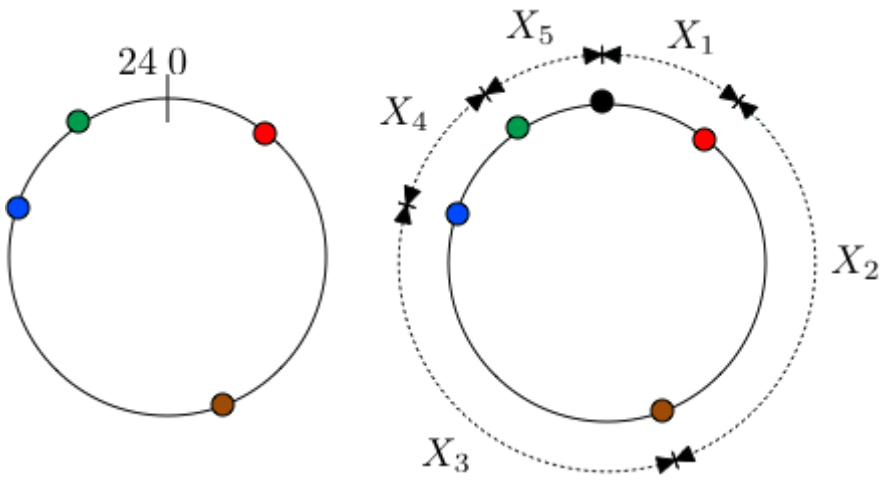

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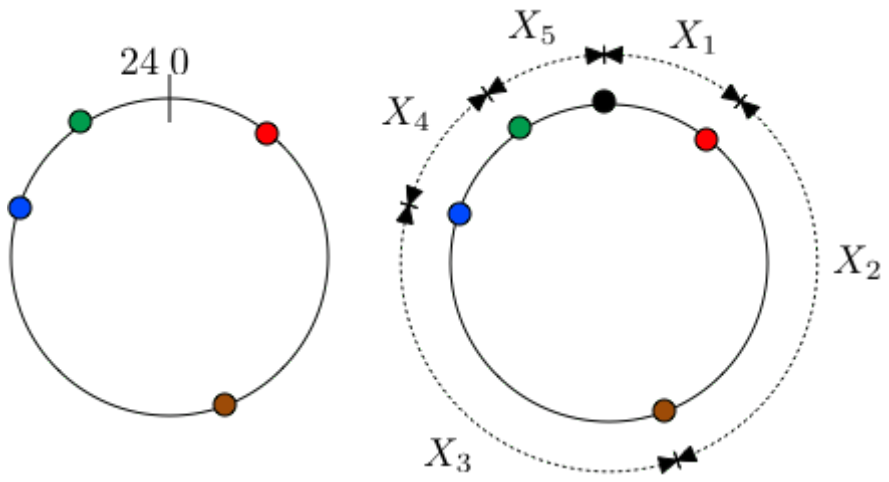

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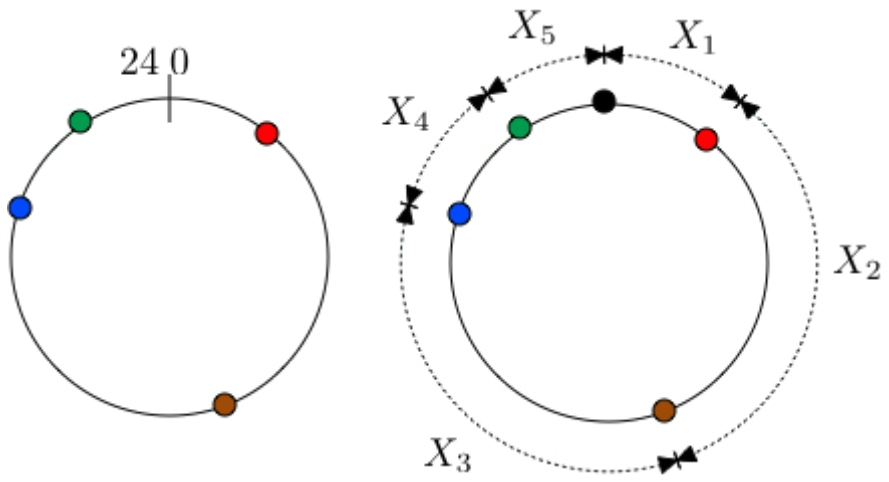

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$$E[X] = E[X|A] Pr[A] + E[X|\bar{A}] Pr[\bar{A}].$$