Independence and Variance
Independence and Variance

1. Expected value of nonnegative integer-valued RV
2. Waiting for a bus
3. A note about expectation
4. Independent random variables.
5. Variance.
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

Expectation: \( E[X] = \sum_a a \times Pr[X = a] = \sum_\omega X(\omega)Pr[\omega]. \)
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

**Expectation:** \( E[X] = \sum_a a \times Pr[X = a] = \sum_\omega X(\omega)Pr[\omega] \).

**Linearity:** \( E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n] \).
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

**Expectation:** \( E[X] = \sum a \times Pr[X = a] = \sum_{\omega} X(\omega) Pr[\omega]. \)

**Linearity:** \( E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]. \)

**Binomial:** \( Pr[X = i] = \binom{n}{k} p^i (1 - p)^{(n-i)}; \)
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

Expectation: \( E[X] = \sum_a a \times Pr[X = a] = \sum_\omega X(\omega)Pr[\omega] \).

Linearity: \( E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n] \).

Binomial: \( Pr[X = i] = \binom{n}{k} p^i (1 - p)^{n-i}; E(X) = pn \).
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

**Expectation:**  
\[ E[X] = \sum_a a \times Pr[X = a] = \sum_\omega X(\omega) Pr[\omega]. \]

**Linearity:**  
\[ E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]. \]

**Binomial:**  
\[ Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}; E(X) = pn. \]

**Geometric:**  
\[ Pr[X = i] = (1 - p)^{i-1} p; \]
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

**Expectation:**
\[ E[X] = \sum_a a \times Pr[X = a] = \sum_\omega X(\omega) Pr[\omega]. \]

**Linearity:**
\[ E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]. \]

**Binomial:**
\[ Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}; E(X) = pn. \]

**Geometric:**
\[ Pr[X = i] = (1 - p)^{i-1} p; E(X) = \frac{1}{p}; \]
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

Expectation: \( E[X] = \sum a \times Pr[X = a] = \sum_\omega X(\omega)Pr[\omega] \).

Linearity: \( E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n] \).

Binomial: \( Pr[X = i] = \binom{n}{k} p^i (1-p)^{n-i}; E(X) = pn \).

Geometric: \( Pr[X = i] = (1-p)^{i-1}p; E(X) = \frac{1}{p}; \) memoryless.
Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

Expectation: \( E[X] = \sum_a a \times Pr[X = a] = \sum_\omega X(\omega)Pr[\omega] \).

Linearity: \( E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]. \)

Binomial: \( Pr[X = i] = \binom{n}{k} p^i (1 - p)^{n-i}; E(X) = pn. \)

Geometric: \( Pr[X = i] = (1 - p)^{(i-1)}p; E(X) = \frac{1}{p}; \) memoryless.

Condition Expectation: \( E[X|A]. \)
Expectation; Conditional Expectation; B(n, p); G(p)

Expectation:  
\[ E[X] = \sum a \times Pr[X = a] = \sum_{\omega} X(\omega)Pr[\omega]. \]

Linearity:  
\[ E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]. \]

Binomial:  
\[ Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}; E(X) = np. \]

Geometric:  
\[ Pr[X = i] = (1-p)^{(i-1)}p; E(X) = \frac{1}{p}; \text{ memoryless.} \]

Condition Expectation:  
\[ E[X|A]. \text{ Linear and} \]
\[ E[X] = E[X|A]Pr[A] + E[X|\bar{A}]Pr[\bar{A}]. \]
Expected Value of Nonnegative Integer RV

**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$
Expected Value of Nonnegative Integer RV

**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

**Proof:** One has

$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$

$$= \sum_{i=1}^{\infty} i \{Pr[X \geq i] - Pr[X \geq i + 1]\}$$
Expected Value of Nonnegative Integer RV

**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

**Proof:** One has

$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$

$$= \sum_{i=1}^{\infty} i \{Pr[X \geq i] - Pr[X \geq i + 1]\}$$

$$= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i + 1]\}$$
Expected Value of Nonnegative Integer RV

**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

**Proof:** One has

$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$

$$= \sum_{i=1}^{\infty} i \{Pr[X \geq i] - Pr[X \geq i + 1]\}$$

$$= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i + 1]\}$$

$$= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - (i - 1) \times Pr[X \geq i]\}$$
Expected Value of Nonnegative Integer RV

**Theorem:** For a r.v. $X$ that takes values in $\{0, 1, 2, \ldots\}$, one has

$$E[X] = \sum_{i=1}^{\infty} Pr[X \geq i].$$

**Proof:** One has

$$E[X] = \sum_{i=1}^{\infty} i \times Pr[X = i]$$

$$= \sum_{i=1}^{\infty} i\{Pr[X \geq i] - Pr[X \geq i + 1]\}$$

$$= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - i \times Pr[X \geq i + 1]\}$$

$$= \sum_{i=1}^{\infty} \{i \times Pr[X \geq i] - (i - 1) \times Pr[X \geq i]\}$$

$$= \sum_{i=1}^{\infty} Pr[X \geq i].$$
Riding the bus.

$n$ buses arrive uniformly at random throughout a 24 hour day.
Riding the bus.

$n$ buses arrive uniformly at random throughout a 24 hour day. What is the time between buses?
Riding the bus.

$n$ buses arrive uniformly at random throughout a 24 hour day. What is the time between buses? How long do you expect to wait for a bus?
Riding the bus.

$n$ buses arrive uniformly at random throughout a 24 hour day. What is the time between buses? How long do you expect to wait for a bus? Here are typical arrival times, independent and uniform in $[0, 24]$. 

![Bus Diagram]
Riding the bus.

\( n \) buses arrive uniformly at random throughout a 24 hour day. What is the time between buses? How long do you expect to wait for a bus? Here are typical arrival times, independent and uniform in \([0, 24]\).

Here is an alternative picture (left)
Riding the bus.

$n$ buses arrive uniformly at random throughout a 24 hour day. What is the time between buses? How long do you expect to wait for a bus? Here are typical arrival times, independent and uniform in $[0, 24]$.

The $n$ gaps between busses add up to 24. Thus $n$ times the expected gap is 24 and the expected gap between busses is $24/n$. 
Riding the bus.

How about the waiting time until the next bus?
Riding the bus.

How about the waiting time until the next bus? Add the black dot uniformly at random and pretend that it represents your arrival time.
Riding the bus.

How about the waiting time until the next bus?

Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random.
Riding the bus.

How about the waiting time until the next bus?
Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random. Then,

\[ 24 = E[X_1 + \cdots + X_5] \]
Riding the bus.

How about the waiting time until the next bus?
Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random. Then,

$$24 = E[X_1 + \cdots + X_5] = 5E[X_1],$$
Riding the bus.

How about the waiting time until the next bus?
Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random. Then,

\[ 24 = E[X_1 + \cdots + X_5] = 5E[X_1], \text{ by linearity and symmetry.} \]
Riding the bus.

How about the waiting time until the next bus?

Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random. Then,

\[ 24 = E[X_1 + \cdots + X_5] = 5E[X_1], \text{ by linearity and symmetry.} \]

Hence, \( E[X_1] = E[X_m] = \frac{24}{5} = \)
Riding the bus.

How about the waiting time until the next bus?
Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random. Then,

\[ 24 = E[X_1 + \cdots + X_5] = 5E[X_1], \] by linearity and symmetry.

Hence, \[ E[X_1] = E[X_m] = 24/5 = 24/(n+1) \] for \( n \) busses.
Riding the bus.

How about the waiting time until the next bus?
Add the black dot uniformly at random and pretend that it represents your arrival time. This is legitimate, because given the black dot, the other dots are uniform at random. Then,

\[ 24 = E[X_1 + \cdots + X_5] = 5E[X_1], \] by linearity and symmetry.

Hence, \[ E[X_1] = E[X_m] = 24/5 = 24/(n+1) \] for \( n \) busses. Note the paradox: the average time between the previous and next bus is \( 48/(n+1) > 24/n \).
A note about expectation

Recall that

\[
E[X] = \sum_x x \ Pr[X = x] = \sum_\omega X(\omega) Pr[\omega]. 
\]  
(1)
A note about expectation

Recall that

\[ E[X] = \sum_{x} x \Pr[X = x] = \sum_{\omega} X(\omega)Pr[\omega]. \]  \hspace{1cm} (1)

**Theorem**
(a) \( E[g(X))] = \sum_{x} g(x)Pr[X = x]. \)
A note about expectation

Recall that

\[ E[X] = \sum_x x \ Pr[X = x] = \sum_\omega X(\omega) Pr[\omega]. \]  \hspace{1cm} (1)

**Theorem**

(a) \( E[g(X))] = \sum_x g(x) Pr[X = x]. \)

(b) \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z]. \)
A note about expectation

Recall that

\[ E[X] = \sum_x x \Pr[X = x] = \sum_\omega X(\omega)Pr[\omega]. \tag{1} \]

**Theorem**

(a) \( E[g(X)] = \sum_x g(x) \Pr[X = x] \).

(b) \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) \Pr[X = x, Y = y, Z = z] \).

**Proof:**

(a) One has
A note about expectation

Recall that

\[ E[X] = \sum_{x} x \ Pr[X = x] = \sum_{\omega} X(\omega)Pr[\omega]. \quad (1) \]

**Theorem**

(a) \( E[g(X))] = \sum_{x} g(x)Pr[X = x]. \)

(b) \( E[g(X, Y, Z)] = \sum_{x, y, z} g(x, y, z)Pr[X = x, Y = y, Z = z]. \)

**Proof:**

(a) One has

\[ E[g(X)] = \sum_{\omega} g(X(\omega))Pr[\omega], \]
A note about expectation

Recall that

\[ E[X] = \sum_x x \Pr[X = x] = \sum_{\omega} X(\omega) \Pr[\omega]. \]  

(1)

**Theorem**

(a) \( E[g(X)]) = \sum_x g(x) \Pr[X = x] \).

(b) \( E[g(X, Y, Z)] = \sum_{x, y, z} g(x, y, z) \Pr[X = x, Y = y, Z = z] \).

**Proof:**

(a) One has

\[ E[g(X)] = \sum_{\omega} g(X(\omega)) \Pr[\omega], \]  

by (1) applied to \( Y(\omega) = g(X(\omega)) \)
A note about expectation

Recall that

\[ E[X] = \sum_x x \Pr[X = x] = \sum_{\omega} X(\omega) \Pr[\omega]. \quad (1) \]

**Theorem**

(a) \( E[g(X))] = \sum_x g(x) \Pr[X = x]. \)

(b) \( E[g(X, Y, Z)] = \sum_{x, y, z} g(x, y, z) \Pr[X = x, Y = y, Z = z]. \)

**Proof:**

(a) One has

\[
E[g(X)] = \sum_{\omega} g(X(\omega)) \Pr[\omega], \text{ by (1) applied to } Y(\omega) = g(X(\omega)) \\
= \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) \Pr[\omega]
\]
A note about expectation

Recall that

\[ E[X] = \sum_x x \Pr[X = x] = \sum_\omega X(\omega)Pr[\omega]. \tag{1} \]

**Theorem**

(a) \( E[g(X))] = \sum_x g(x)Pr[X = x]. \)

(b) \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z)Pr[X = x, Y = y, Z = z]. \)

**Proof:**

(a) One has

\[ E[g(X)] = \sum_\omega g(X(\omega))Pr[\omega], \text{ by (1) applied to } Y(\omega) = g(X(\omega)) \]

\[ = \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega))Pr[\omega] \]

\[ = \sum_x \sum_{\omega \in X^{-1}(x)} g(x)Pr[\omega] \]
A note about expectation

Recall that

\[ E[X] = \sum_x x \Pr[X = x] = \sum_\omega X(\omega) \Pr[\omega]. \quad (1) \]

**Theorem**

(a) \( E[g(X)) = \sum_x g(x) \Pr[X = x] \).

(b) \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) \Pr[X = x, Y = y, Z = z] \).

**Proof:**

(a) One has

\[
E[g(X)] = \sum_\omega g(X(\omega)) \Pr[\omega], \text{ by (1) applied to } Y(\omega) = g(X(\omega))
\]

\[
= \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega)) \Pr[\omega]
\]

\[
= \sum_x \sum_{\omega \in X^{-1}(x)} g(x) \Pr[\omega] = \sum_x g(x) \Pr[X = x].
\]
A note about expectation

Recall that

\[ E[X] = \sum_x x \Pr[X = x] = \sum_\omega X(\omega)\Pr[\omega]. \]  \hspace{1cm} (1)

**Theorem**

(a) \( E[g(X)] = \sum_x g(x)\Pr[X = x]. \)

(b) \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z)\Pr[X = x, Y = y, Z = z]. \)

**Proof:**

(a) One has

\[
E[g(X)] = \sum_\omega g(X(\omega))\Pr[\omega], \text{ by } (1) \text{ applied to } Y(\omega) = g(X(\omega))
\]

\[
= \sum_x \sum_{\omega \in X^{-1}(x)} g(X(\omega))\Pr[\omega]
\]

\[
= \sum_x \sum_{x \in X^{-1}(x)} g(x)\Pr[\omega] = \sum_x g(x)\Pr[X = x].
\]

(b) is similar.
Independent Random Variables.

Definition: Independence
Independent Random Variables.

**Definition:** Independence

The random variables $X$ and $Y$ are independent if and only if

$$Pr[Y = b|X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$
Independent Random Variables.

**Definition:** Independence

The random variables $X$ and $Y$ are **independent** if and only if

$$Pr[Y = b | X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$
Independent Random Variables.

**Definition:** Independence

The random variables $X$ and $Y$ are **independent** if and only if

$$Pr[Y = b | X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$

**Fact:**

$X, Y$ are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a] Pr[Y = b], \text{ for all } a \text{ and } b.$$
Independent Random Variables.

**Definition:** Independence

The random variables $X$ and $Y$ are **independent** if and only if

$$Pr[Y = b|X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$ 

**Fact:**

$X, Y$ are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b], \text{ for all } a \text{ and } b.$$ 

Indeed:

$$Pr[Y = b|X = a] = \frac{Pr[X = a, Y = b]}{Pr[X = a]} = Pr[Y = b]$$
Independent Random Variables.

**Definition:** Independence

The random variables $X$ and $Y$ are **independent** if and only if

$$Pr[Y = b|X = a] = Pr[Y = b], \text{ for all } a \text{ and } b.$$ 

**Fact:**

$X$, $Y$ are independent if and only if

$$Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b], \text{ for all } a \text{ and } b.$$ 

**Indeed:**

$$Pr[Y = b|X = a] = \frac{Pr[X = a, Y = b]}{Pr[X = a]} = Pr[Y = b]$$

$$\iff Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b].$$
Independence: Examples

**Example 1**
Roll two dices. $X, Y =$ number of pips on the two dice. $X, Y$ are independent.

Indeed:
$$
\Pr[X = a, Y = b] = \frac{1}{36}, \quad \Pr[X = a] = \Pr[Y = b] = \frac{1}{6}.
$$
Independence: Examples

Example 1
Roll two dices. \(X, Y\) = number of pips on the two dice. \(X, Y\) are independent.
Indeed: \(Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}\).
Independence: Examples

Example 1
Roll two dices. $X, Y =$ number of pips on the two dice. $X, Y$ are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two dices. $X =$ total number of pips, $Y =$ number of pips on die 1 minus number on die 2. $X$ and $Y$ are
Independence: Examples

Example 1
Roll two dices. \( X, Y = \) number of pips on the two dice. \( X, Y \) are independent.

Indeed: \( Pr[X = a, Y = b] = \frac{1}{36} \), \( Pr[X = a] = Pr[Y = b] = \frac{1}{6} \).

Example 2
Roll two dices. \( X = \) total number of pips, \( Y = \) number of pips on die 1 minus number on die 2. \( X \) and \( Y \) are not independent.
Independence: Examples

Example 1
Roll two dices. $X, Y =$ number of pips on the two dice. $X, Y$ are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two dices. $X =$ total number of pips, $Y =$ number of pips on die 1 minus number on die 2. $X$ and $Y$ are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12] Pr[Y = 1] > 0$. 
Independence: Examples

Example 1
Roll two dices. $X$, $Y =$ number of pips on the two dice. $X$, $Y$ are independent.

Indeed: $\Pr[X = a, Y = b] = \frac{1}{36}, \Pr[X = a] = \Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two dices. $X =$ total number of pips, $Y =$ number of pips on die 1 minus number on die 2. $X$ and $Y$ are not independent.

Indeed: $\Pr[X = 12, Y = 1] = 0 \neq \Pr[X = 12] \Pr[Y = 1] > 0$.

Example 3
Flip a fair coin five times, $X =$ number of $H$s in first three flips, $Y =$ number of $H$s in last two flips. $X$ and $Y$ are independent.
Independence: Examples

Example 1
Roll two dices. \( X, Y = \) number of pips on the two dice. \( X, Y \) are independent.

Indeed: \( \Pr[X = a, Y = b] = \frac{1}{36}, \Pr[X = a] = \Pr[Y = b] = \frac{1}{6}. \)

Example 2
Roll two dices. \( X = \) total number of pips, \( Y = \) number of pips on die 1 minus number on die 2. \( X \) and \( Y \) are not independent.

Indeed: \( \Pr[X = 12, Y = 1] = 0 \neq \Pr[X = 12] \Pr[Y = 1] > 0. \)

Example 3
Flip a fair coin five times, \( X = \) number of \( H \)s in first three flips, \( Y = \) number of \( H \)s in last two flips. \( X \) and \( Y \) are independent.

Indeed:

\[
\Pr[X = a, Y = b] = \binom{3}{a} \binom{2}{b} 2^{-5}
\]
Independence: Examples

Example 1
Roll two dices. $X, Y =$ number of pips on the two dice. $X, Y$ are independent.

Indeed: $Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6}$.

Example 2
Roll two dices. $X =$ total number of pips, $Y =$ number of pips on die 1 minus number on die 2. $X$ and $Y$ are not independent.

Indeed: $Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0$.

Example 3
Flip a fair coin five times, $X =$ number of $H$s in first three flips, $Y =$ number of $H$s in last two flips. $X$ and $Y$ are independent.

Indeed:

$$Pr[X = a, Y = b] = \binom{3}{a} \binom{2}{b} 2^{-5} = \binom{3}{a} 2^{-3} \times \binom{2}{b} 2^{-2}$$
Independence: Examples

Example 1
Roll two dices. \( X, Y \) = number of pips on the two dice. \( X, Y \) are independent.

Indeed: \( Pr[X = a, Y = b] = \frac{1}{36}, Pr[X = a] = Pr[Y = b] = \frac{1}{6} \).

Example 2
Roll two dices. \( X \) = total number of pips, \( Y \) = number of pips on die 1 minus number on die 2. \( X \) and \( Y \) are not independent.

Indeed: \( Pr[X = 12, Y = 1] = 0 \neq Pr[X = 12]Pr[Y = 1] > 0 \).

Example 3
Flip a fair coin five times, \( X \) = number of Hs in first three flips, \( Y \) = number of Hs in last two flips. \( X \) and \( Y \) are independent.

Indeed:

\[
Pr[X = a, Y = b] = \binom{3}{a} \binom{2}{b} 2^{-5} = \binom{3}{a} 2^{-3} \times \binom{2}{b} 2^{-2} = Pr[X = a]Pr[Y = b].
\]
A useful observation about independence

**Theorem**

\[ X \text{ and } Y \text{ are independent if and only if } \Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \text{ for all } A, B \subset \mathbb{R}. \]

**Proof:**

If (\( \Leftarrow \)):

Choose \( A = \{a\} \) and \( B = \{b\} \).

This shows that \( \Pr[X = a, Y = b] = \Pr[X = a] \Pr[Y = b] \).

Only if (\( \Rightarrow \)):

\[
\Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} \Pr[X = a, Y = b] = \sum_{a \in A} \left( \sum_{b \in B} \Pr[X = a, Y = b] \right) = \sum_{a \in A} \Pr[X = a] \left( \sum_{b \in B} \Pr[Y = b] \right) = \Pr[X \in A] \Pr[Y \in B].
\]
A useful observation about independence

Theorem

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$. 
A useful observation about independence

**Theorem**

$X$ and $Y$ are independent if and only if

$$\Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$.

**Proof:**

If ($\Leftarrow$):

Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $\Pr[X = a, Y = b] = \Pr[X = a] \Pr[Y = b]$.
A useful observation about independence

**Theorem**

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$.

**Proof:**

If $(\Leftarrow)$: Choose $A = \{a\}$ and $B = \{b\}$.
A useful observation about independence

Theorem

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$.

Proof:
If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$. 
A useful observation about independence

Theorem

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$.

Proof:

If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$.

Only if ($\Rightarrow$):
A useful observation about independence

**Theorem**

$X$ and $Y$ are independent if and only if

$$\Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \text{ for all } A, B \subset \mathbb{R}.$$ 

**Proof:**

If $(\iff)$: Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $\Pr[X = a, Y = b] = \Pr[X = a] \Pr[Y = b]$.

Only if $(\implies)$:

$$\Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} \Pr[X = a, Y = b]$$
A useful observation about independence

**Theorem**

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \text{ for all } A, B \subset \mathbb{R}.$$ 

**Proof:**

If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$.

Only if ($\Rightarrow$):

$$Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b].$$
A useful observation about independence

Theorem

$X$ and $Y$ are independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \text{ for all } A, B \subset \mathbb{R}.$$  

Proof:

If $(\Leftarrow)$: Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$.

Only if $(\Rightarrow)$:

$$Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b]$$

$$= \sum_{a \in A} [\sum_{b \in B} Pr[X = a]Pr[Y = b]]$$
A useful observation about independence

**Theorem**

$X$ and $Y$ are independent if and only if

\[
Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \text{ for all } A, B \subset \mathbb{R}.
\]

**Proof:**

If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]$.

Only if ($\Rightarrow$):

\[
Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b] = \sum_{a \in A} \left[ \sum_{b \in B} Pr(X = a)Pr(Y = b) \right] = \sum_{a \in A} Pr[X = a] \sum_{b \in B} Pr(Y = b).
\]
A useful observation about independence

**Theorem**

$X$ and $Y$ are independent if and only if

$$\Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B]$$

for all $A, B \subset \mathbb{R}$.

**Proof:**

If ($\Leftarrow$): Choose $A = \{a\}$ and $B = \{b\}$.

This shows that $\Pr[X = a, Y = b] = \Pr[X = a] \Pr[Y = b]$.

Only if ($\Rightarrow$):

\[
\Pr[X \in A, Y \in B] = \sum_{a \in A} \sum_{b \in B} \Pr[X = a, Y = b] = \sum_{a \in A} \sum_{b \in B} \Pr[X = a] \Pr[Y = b]
\]

\[
= \sum_{a \in A} \left[ \sum_{b \in B} \Pr[X = a] \Pr[Y = b] \right] = \sum_{a \in A} \Pr[X = a] \left[ \sum_{b \in B} \Pr[Y = b] \right]
\]

\[
= \sum_{a \in A} \Pr[X = a] \Pr[Y \in B]
\]
A useful observation about independence

**Theorem**

X and Y are independent if and only if

\[ Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \text{ for all } A, B \subset \mathbb{R}. \]

**Proof:**

If (\(\iff\)): Choose \(A = \{a\}\) and \(B = \{b\}\).

This shows that \(Pr[X = a, Y = b] = Pr[X = a]Pr[Y = b]\).

Only if (\(\implies\)):

\[
Pr[X \in A, Y \in B]
= \sum_{a \in A} \sum_{b \in B} Pr[X = a, Y = b]
= \sum_{a \in A} \sum_{b \in B} Pr[X = a]Pr[Y = b]
= \sum_{a \in A} \left[ \sum_{b \in B} Pr[X = a]Pr[Y = b] \right]
= \sum_{a \in A} Pr[X = a]Pr[Y \in B] = Pr[X \in A]Pr[Y \in B].
\]

\(\square\)
Functions of Independent random Variables

**Theorem** Functions of independent RVs are independent

Let $X, Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$
**Theorem** Functions of independent RVs are independent

Let $X$, $Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

**Proof:**
Recall the definition of inverse image:
Functions of Independent random Variables

**Theorem** Functions of independent RVs are independent
Let $X, Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

**Proof:**
Recall the definition of inverse image:

$$h(z) \in C \iff z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \quad (2)$$
Functions of Independent random Variables

**Theorem** Functions of independent RVs are independent
Let $X$, $Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

**Proof:**
Recall the definition of inverse image:

$$h(z) \in C \iff z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \quad (2)$$

Now,

$$\Pr[f(X) \in A, g(Y) \in B] = \Pr[X \in f^{-1}(A), Y \in g^{-1}(B)], \text{ by } (2)$$
Functions of Independent random Variables

**Theorem** Functions of independent RVs are independent
Let $X, Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

**Proof:**
Recall the definition of inverse image:

$$h(z) \in C \iff z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \quad (2)$$

Now,

$$Pr[f(X) \in A, g(Y) \in B]$$

$$= Pr[X \in f^{-1}(A), Y \in g^{-1}(B)], \text{ by } (2)$$

$$= Pr[X \in f^{-1}(A)] Pr[Y \in g^{-1}(B)],$$
Functions of Independent random Variables

**Theorem** Functions of independent RVs are independent
Let $X, Y$ be independent RV. Then

$$f(X)\text{ and } g(Y)\text{ are independent, for all } f(\cdot), g(\cdot).$$

**Proof:**
Recall the definition of inverse image:

$$h(z) \in C \iff z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \quad (2)$$

Now,

$$Pr[f(X) \in A, g(Y) \in B]$$
$$= Pr[X \in f^{-1}(A), Y \in g^{-1}(B)], \text{ by (2)}$$
$$= Pr[X \in f^{-1}(A)]Pr[Y \in g^{-1}(B)], \text{ since } X, Y \text{ ind.}$$
$$= Pr[f(X) \in A]Pr[g(Y) \in B],$$
**Functions of Independent random Variables**

**Theorem** Functions of independent RVs are independent

Let $X$, $Y$ be independent RV. Then

$$f(X) \text{ and } g(Y) \text{ are independent, for all } f(\cdot), g(\cdot).$$

**Proof:**

Recall the definition of inverse image:

$$h(z) \in C \Leftrightarrow z \in h^{-1}(C) := \{z \mid h(z) \in C\}. \quad (2)$$

Now,

$$Pr[f(X) \in A, g(Y) \in B]$$

$$= Pr[X \in f^{-1}(A), Y \in g^{-1}(B)], \text{ by (2)}$$

$$= Pr[X \in f^{-1}(A)]Pr[Y \in g^{-1}(B)], \text{ since } X, Y \text{ ind.}$$

$$= Pr[f(X) \in A]Pr[g(Y) \in B], \text{ by (2)}. $$
Theorem
Let $X, Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$
Mean of product of independent RV

**Theorem**
Let $X, Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) \Pr[X = x, Y = y]$. 
Mean of product of independent RV

**Theorem**
Let $X$, $Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x,y} xy Pr[X = x, Y = y]$$
Mean of product of independent RV

**Theorem**
Let $X, Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x,y} xy Pr[X = x, Y = y] = \sum_{x,y} xy Pr[X = x] Pr[Y = y].$$
Mean of product of independent RV

**Theorem**
Let $X$, $Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x,y} xy Pr[X = x, Y = y] = \sum_{x,y} xy Pr[X = x] Pr[Y = y], \text{ by ind.}$$

$$= \sum_x [\sum_y xy Pr[X = x] Pr[Y = y]]$$
Mean of product of independent RV

Theorem
Let $X, Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

Proof:
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x,y} xy Pr[X = x, Y = y] = \sum_{x,y} xy Pr[X = x] Pr[Y = y], \text{ by ind.}$$

$$= \sum_{x} \sum_{y} xy Pr[X = x] Pr[Y = y] = \sum_{x} \left[ x Pr[X = x] \left( \sum_{y} y Pr[Y = y] \right) \right]$$
Mean of product of independent RV

**Theorem**
Let $X$, $Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x,y} xyPr[X = x, Y = y] = \sum_{x,y} xyPr[X = x]Pr[Y = y], \text{ by ind.}$$

$$= \sum_x \left( \sum_y xyPr[X = x]Pr[Y = y] \right) = \sum_x [xPr[X = x](\sum_y yPr[Y = y])]$$

$$= \sum_x [xPr[X = x]E[Y]]$$
Mean of product of independent RV

**Theorem**
Let $X, Y$ be independent RVs. Then

$$E[XY] = E[X]E[Y].$$

**Proof:**
Recall that $E[g(X, Y)] = \sum_{x,y} g(x, y) Pr[X = x, Y = y]$. Hence,

$$E[XY] = \sum_{x,y} xy Pr[X = x, Y = y] = \sum_{x,y} xy Pr[X = x] Pr[Y = y], \text{ by ind.}$$

$$= \sum_{x} \left( \sum_{y} xy Pr[X = x] Pr[Y = y] \right) = \sum_{x} \left( x Pr[X = x] \left( \sum_{y} y Pr[Y = y] \right) \right)$$

$$= \sum_{x} \left( x Pr[X = x] E[Y] \right) = E[X]E[Y].$$
Examples


Examples


$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$
Examples


$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$
$$= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0$$
Examples


$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$
$$= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0$$
$$= 14.$$
Examples


Then

$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$

$$= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0$$

$$= 14.$$ 

(2) Let $X, Y$ be independent and $U[1, 2, \ldots n]$. Then
Examples


$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$
$$= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0$$
$$= 14.$$ 

(2) Let $X, Y$ be independent and $U[1, 2, \ldots, n]$. Then

Examples


$$E[(X + 2Y + 3Z)^2] = E[X^2 + 4Y^2 + 9Z^2 + 4XY + 12YZ + 6XZ]$$
$$= 1 + 4 + 9 + 4 \times 0 + 12 \times 0 + 6 \times 0$$
$$= 14.$$ 

(2) Let $X, Y$ be independent and $U[1, 2, \ldots n]$. Then

$$= \frac{1 + 3n + 2n^2}{3} - \frac{(n+1)^2}{2}.$$
Mutually Independent Random Variables

Definition

(a) $X, Y, Z$ are mutually independent if
$$\Pr[X = x, Y = y, Z = z] = \Pr[X = x] \Pr[Y = y] \Pr[Z = z],$$
for all $x, y, z$.

(b) $\{X_j, j \in J\}$ are mutually dependent if
$$\Pr[X_k = x_k, k \in K] = \prod_{k \in K} \Pr[X_k = x_k],$$
for all finite $K \subset J$ and all $x_k$.

Note: If $A, B, C$ are pairwise independent but not mutually independent, then $X = 1_A, Y = 1_B, Z = 1_C$ are pairwise independent but not mutually independent.
Mutually Independent Random Variables

Definition

(a) $X, Y, Z$ are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$
Mutually Independent Random Variables

Definition

(a) $X, Y, Z$ are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$

(b) $\{X_j, j \in J\}$ are mutually dependent if

$$Pr[X_k = x_k, k \in K] = \prod_{k \in K} Pr[X_k = x_k], \text{ for all finite } K \subset J \text{ and all } x_k.$$
Mutually Independent Random Variables

**Definition**

(a) $X, Y, Z$ are mutually independent if

$$Pr[X = x, Y = y, Z = z] = Pr[X = x]Pr[Y = y]Pr[Z = z], \text{ for all } x, y, z.$$ 

(b) $\{X_j, j \in J\}$ are mutually dependent if

$$Pr[X_k = x_k, k \in K] = \prod_{k \in K} Pr[X_k = x_k], \text{ for all finite } K \subset J \text{ and all } x_k.$$ 

Note: If $A, B, C$ are pairwise independent but not mutually independent, then $X = 1_A, Y = 1_B, Z = 1_C$ are pairwise independent but not mutually independent.
Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y,Z) \text{ are not independent.}$$
Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

**Example:** Flip two fair coins, $X = 1\{\text{coin 1 is } H\}$, $Y = 1\{\text{coin 2 is } H\}$, $Z = X \oplus Y$. 
Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

\[ f(X) \text{ and } g(Y, Z) \text{ are not independent.} \]

**Example:** Flip two fair coins,
$X = 1\{\text{coin 1 is } H\}$, $Y = 1\{\text{coin 2 is } H\}$, $Z = X \oplus Y$. Then, $X, Y, Z$ are pairwise independent.
Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

**Example:** Flip two fair coins, $X = 1\{\text{coin 1 is } H\}, Y = 1\{\text{coin 2 is } H\}, Z = X \oplus Y$. Then, $X, Y, Z$ are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. 

Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

**Example:** Flip two fair coins, $X = 1 \{\text{coin 1 is } H\}$, $Y = 1 \{\text{coin 2 is } H\}$, $Z = X \oplus Y$. Then, $X, Y, Z$ are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of $X$. 
Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

**Example:** Flip two fair coins, $X = 1\{\text{coin 1 is } H\}, Y = 1\{\text{coin 2 is } H\}, Z = X \oplus Y$. Then, $X, Y, Z$ are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of $X$.

**Theorem**

Let $\{X_j, j \in J\}$ be mutually independent RVs. Then

$$f_1(X_j, j \in K_1), f_2(X_j, j \in K_2), \ldots \text{ are mutually independent}$$

for all pairwise disjoint $K_m$ and functions $f_m(\cdot)$. 
Functions of mutually independent RVs

If $X, Y, Z$ are pairwise independent, but not mutually independent, it may be that

$$f(X) \text{ and } g(Y, Z) \text{ are not independent.}$$

Example: Flip two fair coins, $X = 1\{\text{coin 1 is } H\}, Y = 1\{\text{coin 2 is } H\}, Z = X \oplus Y$. Then, $X, Y, Z$ are pairwise independent. Let $g(Y, Z) = Y \oplus Z$. Then $g(Y, Z) = X$ is not independent of $X$.

**Theorem**

Let $\{X_j, j \in J\}$ be mutually independent RVs. Then

$$f_1(X_j, j \in K_1), f_2(X_j, j \in K_2), \ldots \text{ are mutually independent}$$

for all pairwise disjoint $K_m$ and functions $f_m(\cdot)$.

**Proof:**

Try it, you will like it.
Variance

The variance measures the deviation from the mean value.

Definition:

The variance of $X$ is $\sigma^2(X) := \text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

$\sigma(X)$ is called the standard deviation of $X$. 

![Graph showing the distribution of variance with Var = 1 and Var = 10]
Variance

The variance measures the deviation from the mean value.

\[
\text{The variance of } X \text{ is } \sigma^2(X) := \text{var}[X] = E[(X - E[X])^2].
\]

\[
\sigma(X) \text{ is called the standard deviation of } X.
\]
Variance

The variance measures the deviation from the mean value.

**Definition:** The variance of $X$ is
Variance

The variance measures the deviation from the mean value.

**Definition:** The variance of $X$ is

$$\sigma^2(X) := \text{var}[X] = E[(X - E[X])^2].$$
Variance

The variance measures the deviation from the mean value.

**Definition:** The variance of $X$ is

$$
\sigma^2(X) := \text{var}[X] = E[(X - E[X])^2].
$$

$\sigma(X)$ is called the standard deviation of $X$. 
Variance

The variance measures the deviation from the mean value.

**Definition:** The variance of $X$ is

$$\sigma^2(X) := var[X] = E[(X - E[X])^2].$$

$\sigma(X)$ is called the standard deviation of $X$. 
Variance and Standard Deviation

Fact:

Variance and Standard Deviation

Fact:

\[
\]

Indeed:

\[
\text{var}(X) = E[(X - E[X])^2]
\]
Variance and Standard Deviation

Fact:


Indeed:

$$\text{var}(X) = E[(X - E[X])^2]$$

$$= E[X^2 - 2XE[X] + E[X]^2]$$
Fact:

\[ \text{var}[X] = E[X^2] - E[X]^2. \]

Indeed:

\[
\begin{align*}
\text{var}(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - 2E[X]E[X] + E[X]^2,
\end{align*}
\]
Variance and Standard Deviation

**Fact:**

\[ \text{var}[X] = E[X^2] - E[X]^2. \]

**Indeed:**

\[
\text{var}(X) = \text{E}[(X - E[X])^2]
\]
\[
= E[X^2 - 2XE[X] + E[X]^2]
\]
\[
= E[X^2] - 2E[X]E[X] + E[X]^2, \text{ by linearity}
\]
Fact:

\[ \text{var}[X] = E[X^2] - E[X]^2. \]

Indeed:

\[
\begin{align*}
\text{var}(X) &= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - 2E[X]E[X] + E[X]^2, \text{ by linearity} \\
&= E[X^2] - E[X]^2.
\end{align*}
\]
A simple example

This example illustrates the term ‘standard deviation.’
A simple example

This example illustrates the term ‘standard deviation.’
A simple example

This example illustrates the term ‘standard deviation.’

Consider the random variable $X$ such that

$$X = \begin{cases} 
\mu - \sigma, & \text{w.p. } 1/2 \\
\mu + \sigma, & \text{w.p. } 1/2.
\end{cases}$$
A simple example

This example illustrates the term ‘standard deviation.’

Consider the random variable $X$ such that

$$X = \begin{cases} 
\mu - \sigma, & \text{w.p. } 1/2 \\
\mu + \sigma, & \text{w.p. } 1/2.
\end{cases}$$

Then, $E[X] = \mu$ and $(X - E[X])^2 = \sigma^2$. 
A simple example

This example illustrates the term ‘standard deviation.’

Consider the random variable $X$ such that

$$X = \begin{cases} 
\mu - \sigma, & \text{w.p. } 1/2 \\
\mu + \sigma, & \text{w.p. } 1/2.
\end{cases}$$

Then, $E[X] = \mu$ and $(X - E[X])^2 = \sigma^2$. Hence,

$$\text{var}(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$
Example

Consider $X$ with

$$X = \begin{cases} -1, & \text{w. p. 0.99} \\ 99, & \text{w. p. 0.01}. \end{cases}$$
Example

Consider $X$ with

$$X = \begin{cases} -1, & \text{w. p. 0.99} \\ 99, & \text{w. p. 0.01}. \end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$
Example

Consider $X$ with

$$X = \begin{cases} 
-1, & \text{w. p. 0.99} \\
99, & \text{w. p. 0.01}.
\end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$
$$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$
Example

Consider $X$ with

$$X = \begin{cases} 
-1, & \text{w. p. 0.99} \\
99, & \text{w. p. 0.01.}
\end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$  

$$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$  

$$Var(X) \approx 100 \implies \sigma(X) \approx 10.$$
Consider $X$ with

$$X = \begin{cases} -1, & \text{w. p. 0.99} \\ 99, & \text{w. p. 0.01}. \end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$ 

$$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$ 

$$Var(X) \approx 100 \implies \sigma(X) \approx 10.$$ 

Also,

$$E(|X|) = 1 \times 0.99 + 99 \times 0.01 = 1.98.$$
Example

Consider $X$ with

$$X = \begin{cases} 
-1, & \text{w. p. 0.99} \\
99, & \text{w. p. 0.01}.
\end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$ 
$$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$ 
$$Var(X) \approx 100 \implies \sigma(X) \approx 10.$$ 

Also,

$$E(|X|) = 1 \times 0.99 + 99 \times 0.01 = 1.98.$$ 

Thus, $\sigma(X) \neq E[|X - E[X]|]$!
Example

Consider $X$ with

\[ X = \begin{cases} 
-1, & \text{w. p. 0.99} \\
99, & \text{w. p. 0.01}.
\end{cases} \]

Then

\[
E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.
\]
\[
E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.
\]
\[
Var(X) \approx 100 \implies \sigma(X) \approx 10.
\]

Also,

\[
E(|X|) = 1 \times 0.99 + 99 \times 0.01 = 1.98.
\]

Thus, $\sigma(X) \neq E[|X - E[X]|]$!

Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i$$
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$  

Also,

$$E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2}{12}.$$
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i$$

$$= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$ 

Also,

$$E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2$$

$$= \frac{1 + 3n + 2n^2}{6},$$
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i$$

$$= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Also,

$$E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2$$

$$= \frac{1 + 3n + 2n^2}{6}, \text{ as you can verify.}$$
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$ 

Also,

$$E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1 + 3n + 2n^2}{6}, \text{ as you can verify.}$$

This gives

$$\text{var}(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$
Fixed points.

Number of fixed points in a random permutation of $n$ items.
Fixed points.

Number of fixed points in a random permutation of $n$ items.

“Number of student that get homework back.”
Fixed points.

Number of fixed points in a random permutation of $n$ items. “Number of student that get homework back.”

$$X = X_1 + X_2 \cdots + X_n$$
Fixed points.

Number of fixed points in a random permutation of \( n \) items.
“Number of student that get homework back.”

\[ X = X_1 + X_2 \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.
Fixed points.
Number of fixed points in a random permutation of $n$ items. “Number of student that get homework back.”

$X = X_1 + X_2 \cdots + X_n$

where $X_i$ is indicator variable for $i$th student getting hw back.

$$E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).$$
Fixed points.

Number of fixed points in a random permutation of \( n \) items. “Number of student that get homework back.”

\[ X = X_1 + X_2 \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.

\[
E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).
\]

\[
= +
\]

\[
E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]
\]
Fixed points.

Number of fixed points in a random permutation of \( n \) items.

“Number of student that get homework back.”

\[ X = X_1 + X_2 \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.

\[
E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).
\]

\[
= \frac{1}{n} + \frac{1}{n} = 2.
\]

\[
E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0] = \frac{1}{n}
\]
Fixed points.

Number of fixed points in a random permutation of \( n \) items. “Number of student that get homework back.”

\[ X = X_1 + X_2 + \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.

\[
E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_i X_j).
\]

\[
= \frac{1}{n} + \frac{1}{n(n-1)}
\]

\[
E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]
\]

\[
= \frac{1}{n}
\]
Fixed points.

Number of fixed points in a random permutation of $n$ items. “Number of student that get homework back.”

$$X = X_1 + X_2 \cdots + X_n$$

where $X_i$ is indicator variable for $i$th student getting hw back.

$$E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).$$

$$= n \times \frac{1}{n} + \ldots$$

$$E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]$$

$$= \frac{1}{n}$$
Fixed points.

Number of fixed points in a random permutation of $n$ items. “Number of student that get homework back.”

$X = X_1 + X_2 \cdots + X_n$

where $X_i$ is indicator variable for $i$th student getting hw back.

$$E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).$$

$$= n \times \frac{1}{n} +$$

$$E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]$$

$$= \frac{1}{n}$$

$$E(X_iX_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr["anything else"]$$
Fixed points.

Number of fixed points in a random permutation of $n$ items. “Number of student that get homework back.”

$$X = X_1 + X_2 \cdots + X_n$$

where $X_i$ is indicator variable for $i$th student getting hw back.

$$E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).$$

$$= n \times \frac{1}{n} +$$

$$E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]$$

$$= \frac{1}{n}$$

$$E(X_iX_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr[\text{“anything else”}]$$

$$= \frac{1 \times 1 \times (n-2)!}{n!}$$
Fixed points.

Number of fixed points in a random permutation of \( n \) items.
“Number of student that get homework back.”

\[ X = X_1 + X_2 \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.

\[
E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).
\]

\[
= n \times \frac{1}{n} + n \times \frac{1}{n} + \ldots + n \times \frac{1}{n} = \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = \frac{n}{n} = 1.
\]

\[
E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0] = \frac{1}{n}
\]

\[
E(X_iX_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr[\text{“anything else”}] = \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}
\]
Fixed points.

Number of fixed points in a random permutation of \( n \) items.

“Number of student that get homework back.”

\[ X = X_1 + X_2 \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.

\[
E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).
\]

\[
= n \times \frac{1}{n} + (n)(n-1) \times \frac{1}{n(n-1)}
\]

\[
E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]
= \frac{1}{n}
\]

\[
E(X_iX_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr[“anything else”]
= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}
\]
Fixed points.

Number of fixed points in a random permutation of $n$ items. “Number of student that get homework back.”

$$X = X_1 + X_2 \cdots + X_n$$

where $X_i$ is indicator variable for $i$th student getting hw back.

$$E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).$$

$$= n \times \frac{1}{n} + (n)(n-1) \times \frac{1}{n(n-1)}$$

$$= 1 + 1 = 2.$$

$$E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]$$

$$= \frac{1}{n}$$

$$E(X_iX_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr[“anything else”]$$

$$= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}$$
Fixed points.

Number of fixed points in a random permutation of \( n \) items.

“Number of student that get homework back.”

\[ X = X_1 + X_2 \cdots + X_n \]

where \( X_i \) is indicator variable for \( i \)th student getting hw back.

\[
E(X^2) = \sum_i E(X_i^2) + \sum_{i \neq j} E(X_iX_j).
\]

\[
= n \times \frac{1}{n} + (n)(n-1) \times \frac{1}{n(n-1)}
\]

\[= 1 + 1 = 2. \]

\[
E(X_i^2) = 1 \times Pr[X_i = 1] + 0 \times Pr[X_i = 0]
\]

\[= \frac{1}{n} \]

\[
E(X_iX_j) = 1 \times Pr[X_i = 1 \cap X_j = 1] + 0 \times Pr[“anything else”]
\]

\[= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)} \]

\[Var(X) = E(X^2) - (E(X))^2 = 2 - 1 = 1. \]
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

= Really??!!##...

Too hard!
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

= Really???!!##... 

Too hard!

Ok..
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

= Really???!!##...

Too hard!

Ok.. fine.
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

\[ = \text{Really??!!##...} \]

Too hard!

Ok.. fine.
Let’s do something else.
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

= Really??!!##...

Too hard!

Ok.. fine.
Let’s do something else.
Maybe not much easier...
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

\[ = \text{Really???!!##...} \]

Too hard!
Ok.. fine.
Let’s do something else.
Maybe not much easier...but there is a payoff.
Properties of variance.

1. \( \text{Var}(cX) = c^2 \text{Var}(X) \), where \( c \) is a constant.
Properties of variance.

1. $\text{Var}(cX) = c^2 \text{Var}(X)$, where $c$ is a constant. Scales by $c^2$. 
Properties of variance.

1. \( \text{Var}(cX) = c^2 \text{Var}(X) \), where \( c \) is a constant. Scales by \( c^2 \).
2. \( \text{Var}(X + c) = \text{Var}(X) \), where \( c \) is a constant.
Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where $c$ is a constant. Scales by $c^2$.
2. $Var(X + c) = Var(X)$, where $c$ is a constant. Shifts center.
Properties of variance.

1. $Var(cX) = c^2 Var(X)$, where $c$ is a constant. Scales by $c^2$.
2. $Var(X + c) = Var(X)$, where $c$ is a constant. Shifts center.

Proof:

$$Var(cX) = E((cX)^2) - (E(cX))^2$$
Properties of variance.

1. $\text{Var}(cX) = c^2 \text{Var}(X)$, where $c$ is a constant. Scales by $c^2$.

2. $\text{Var}(X + c) = \text{Var}(X)$, where $c$ is a constant. Shifts center.

Proof:

$$\text{Var}(cX) = E((cX)^2) - (E(cX))^2$$
$$= c^2 E(X^2) - c^2 (E(X))^2$$
Properties of variance.

1. $\text{Var}(cX) = c^2 \text{Var}(X)$, where $c$ is a constant. Scales by $c^2$.

2. $\text{Var}(X + c) = \text{Var}(X)$, where $c$ is a constant. Shifts center.

Proof:

\[
\text{Var}(cX) = E((cX)^2) - (E(cX))^2
= c^2 E(X^2) - c^2 (E(X))^2
= c^2 (E(X^2) - E(X)^2)
\]
Properties of variance.

1. \( \text{Var}(cX) = c^2 \text{Var}(X) \), where \( c \) is a constant.
   Scales by \( c^2 \).

2. \( \text{Var}(X + c) = \text{Var}(X) \), where \( c \) is a constant.
   Shifts center.

Proof:

\[
\text{Var}(cX) = E((cX)^2) - (E(cX))^2 \\
= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
= c^2 \text{Var}(X)
\]
Properties of variance.

1. $\text{Var}(cX) = c^2 \text{Var}(X)$, where $c$ is a constant.
   Scales by $c^2$.

2. $\text{Var}(X + c) = \text{Var}(X)$, where $c$ is a constant.
   Shifts center.

Proof:

\[
\begin{align*}
\text{Var}(cX) & = E((cX)^2) - (E(cX))^2 \\
& = c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
& = c^2 \text{Var}(X) \\
\text{Var}(X + c) & = E((X + c - E(X + c))^2)
\end{align*}
\]
Properties of variance.

1. \( \text{Var}(cX) = c^2 \text{Var}(X) \), where \( c \) is a constant. Scales by \( c^2 \).

2. \( \text{Var}(X + c) = \text{Var}(X) \), where \( c \) is a constant. Shifts center.

Proof:

\[
\text{Var}(cX) = E((cX)^2) - (E(cX))^2 \\
= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
= c^2 \text{Var}(X)
\]

\[
\text{Var}(X + c) = E((X + c - E(X + c))^2) \\
= E((X + c - E(X) - c)^2)
\]
Properties of variance.

1. \(\text{Var}(cX) = c^2 \text{Var}(X)\), where \(c\) is a constant.  
   Scales by \(c^2\).

2. \(\text{Var}(X + c) = \text{Var}(X)\), where \(c\) is a constant.  
   Shifts center.

Proof:

\[
\begin{align*}
\text{Var}(cX) &= E((cX)^2) - (E(cX))^2 \\
&= c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
&= c^2 \text{Var}(X) \\
\text{Var}(X + c) &= E((X + c - E(X + c))^2) \\
&= E((X + c - E(X) - c)^2) \\
&= E((X - E(X))^2)
\end{align*}
\]
Properties of variance.

1. $\text{Var}(cX) = c^2 \text{Var}(X)$, where $c$ is a constant.  
Scales by $c^2$.

2. $\text{Var}(X + c) = \text{Var}(X)$, where $c$ is a constant.  
Shifts center.

Proof:

\[
\begin{align*}
\text{Var}(cX) & = E((cX)^2) - (E(cX))^2 \\
& = c^2 E(X^2) - c^2 (E(X))^2 = c^2 (E(X^2) - E(X)^2) \\
& = c^2 \text{Var}(X)
\end{align*}
\]

\[
\begin{align*}
\text{Var}(X + c) & = E((X + c - E(X + c))^2) \\
& = E((X + c - E(X) - c)^2) \\
& = E((X - E(X))^2) = \text{Var}(X)
\end{align*}
\]
Properties of variance.

1. \( \text{Var}(cX) = c^2 \text{Var}(X) \), where \( c \) is a constant. Scales by \( c^2 \).

2. \( \text{Var}(X + c) = \text{Var}(X) \), where \( c \) is a constant. Shifts center.

Proof:

\[
\text{Var}(cX) = E((cX)^2) - (E(cX))^2
= c^2 E(X^2) - c^2 (E(X))^2
= c^2 (E(X^2) - E(X)^2)
= c^2 \text{Var}(X)
\]

\[
\text{Var}(X + c) = E((X + c - E(X + c))^2)
= E((X + c - E(X) - c)^2)
= E((X - E(X))^2) = \text{Var}(X)
\]
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$Var(X + Y) = Var(X) + Var(Y).$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$. 
Variance of sum of independent random variables

Theorem:
If $X$ and $Y$ are independent, then

$$Var(X + Y) = Var(X) + Var(Y).$$

Proof:
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$\text{var}(X + Y) = E((X + Y)^2)$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$\text{var}(X + Y) = E(((X + Y)^2) = E(X^2 + 2XY + Y^2)$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$\text{var}(X + Y) = E((X + Y)^2) = E(X^2 + 2XY + Y^2)$$

$$= E(X^2) + 2E(XY) + E(Y^2)$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$\text{var}(X + Y) = E((X + Y)^2) = E(X^2 + 2XY + Y^2)$$

$$= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2)$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$Var(X + Y) = Var(X) + Var(Y).$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

$$E(XY) = E(X)E(Y) = 0.$$

Hence,

$$var(X + Y) = E((X + Y)^2) = E(X^2 + 2XY + Y^2)$$

$$= E(X^2) + 2E(XY) + E(Y^2) = E(X^2) + E(Y^2)$$

$$= var(X) + var(Y).$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. 

$E(X_i) = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$

$\text{Var}(X_i) = p - (E(X_i))^2 = p - p^2 = p(1-p)$.

$X = X_1 + X_2 + \ldots + X_n$.

$X_i$ and $X_j$ are independent:

$\Pr[X_i = 1 | X_j = 1] = \Pr[X_i = 1]$.

$\text{Var}(X) = \text{Var}(X_1 + \cdots + X_n) = np(1-p)$. 
Variance of Binomial Distribution.

Flip coin with heads probability $p$.  
$X$- how many heads?
Variance of Binomial Distribution.

Flip coin with heads probability $p$. 
$X$- how many heads?

$$X_i = \begin{cases} 
1 & \text{if } \text{i}th \text{ flip is heads} \\
0 & \text{otherwise}
\end{cases}$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 
1 & \text{if } \text{i\text{th flip is heads}} \\
0 & \text{otherwise}
\end{cases}$$

$$E(X_i^2)$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$.  
$X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p)$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$.  
$X$ - how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } \text{ith flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$

$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$
Variance of Binomial Distribution.

Flip coin with heads probability $p$.  
$X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0$
**Variance of Binomial Distribution.**

Flip coin with heads probability $p$. $X$ - how many heads?

$$X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
0 & \text{otherwise}
\end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1-p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1-p).$$

$p = 0 \implies Var(X_i) = 0$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. 
$X$ - how many heads?

$$X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
0 & \text{otherwise} 
\end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$ 
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies Var(X_i) = 0$

$p = 1$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. 
$X$: how many heads?

$$X_i = \begin{cases} 
1 & \text{if } \text{ith flip is heads} \\
0 & \text{otherwise} 
\end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$ 
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$$p = 0 \implies Var(X_i) = 0$$
$$p = 1 \implies Var(X_i) = 0$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies Var(X_i) = 0$  
$p = 1 \implies Var(X_i) = 0$
Variance of Binomial Distribution.

Flip coin with heads probability \( p \).

\( X \) - how many heads?

\[ X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases} \]

\[ E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p. \]

\[ \text{Var}(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p). \]

\( p = 0 \implies \text{Var}(X_i) = 0 \)

\( p = 1 \implies \text{Var}(X_i) = 0 \)

\( X = X_1 + X_2 + \ldots X_n. \)
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } \text{ith flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$\text{Var}(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies \text{Var}(X_i) = 0$

$p = 1 \implies \text{Var}(X_i) = 0$

$X = X_1 + X_2 + \ldots X_n.$

$X_i$ and $X_j$ are independent:

$$\text{Var}(X) = \text{Var}(X_1 + \cdots + X_n) = \sum_{i=1}^{n} \text{Var}(X_i) = np(1 - p).$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$ - how many heads?

$$X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
0 & \text{otherwise}
\end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies Var(X_i) = 0$  
$p = 1 \implies Var(X_i) = 0$

$X = X_1 + X_2 + \ldots X_n.$

$X_i$ and $X_j$ are independent: $Pr[X_i = 1|X_j = 1] = Pr[X_i = 1]$. 

$p = 0 \implies Var(X_i) = 0$  
$p = 1 \implies Var(X_i) = 0$

$X = X_1 + X_2 + \ldots X_n.$

$X_i$ and $X_j$ are independent: $Pr[X_i = 1|X_j = 1] = Pr[X_i = 1]$. 

$Var(X) = Var(X_1 + \ldots + X_n) = np(1 - p).$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies Var(X_i) = 0$

$p = 1 \implies Var(X_i) = 0$

$X = X_1 + X_2 + \ldots X_n.$

$X_i$ and $X_j$ are independent: $Pr[X_i = 1|X_j = 1] = Pr[X_i = 1].$

$$Var(X) = Var(X_1 + \cdots X_n)$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
0 & \text{otherwise} 
\end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies Var(X_i) = 0$  
$p = 1 \implies Var(X_i) = 0$

$X = X_1 + X_2 + \ldots X_n.$  

$X_i$ and $X_j$ are independent:  
$$Pr[X_i = 1|X_j = 1] = Pr[X_i = 1].$$

$$Var(X) = Var(X_1 + \ldots X_n) = np(1 - p).$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
0 & \text{otherwise}
\end{cases}$$

$$E(X_i^2) = 1^2 \times p + 0^2 \times (1 - p) = p.$$  
$$Var(X_i) = p - (E(X))^2 = p - p^2 = p(1 - p).$$

$p = 0 \implies Var(X_i) = 0$  
$p = 1 \implies Var(X_i) = 0$  

$X = X_1 + X_2 + \ldots X_n$.  

$X_i$ and $X_j$ are independent: $Pr[X_i = 1|X_j = 1] = Pr[X_i = 1]$.  

$$Var(X) = Var(X_1 + \cdots X_n) = np(1 - p).$$
Independence and Variance

\[ E[g(X, Y, Z)] = \sum_{x, y, z} g(x, y, z) \Pr[X = x, Y = y, Z = z] \]

\[ X, Y \text{ independent} \iff \Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \]

\[ \text{Then, } f(X), g(Y) \text{ are independent} \]

\[ E[XY] = E[X]E[Y] \text{ and } \text{var}[X + Y] = \text{var}[X] + \text{var}[Y]. \]

Distributions:

\[ U[1, \ldots, n] : E[X] = \left( \frac{n+1}{2} \right); \text{var}[X] = \left( \frac{n^2-1}{12} \right) \]

\[ B(n, p) : E[X] = np; \text{var}[X] = np(1-p) \]
Summary

Independence and Variance

\[ E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] \]
Summary

**Independence and Variance**

- \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) \Pr[X = x, Y = y, Z = z] \)

- \( X, Y \) independent
  \[ \Leftrightarrow \Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \]
Summary

Independence and Variance

- \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] \)
- \( X, Y \) independent
  \( \Leftrightarrow Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B] \)
- Then, \( f(X), g(Y) \) are independent
Independence and Variance

- \[ E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] \]
- \( X, Y \) independent if \( \iff Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \)
- Then, \( f(X), g(Y) \) are independent
Summary

Independence and Variance

- \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) \Pr[X = x, Y = y, Z = z] \)
- \( X, Y \) independent \( \iff \Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \)
- Then, \( f(X), g(Y) \) are independent
- Also, \( E[XY] = E[X]E[Y] \) and \( \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \).

Distributions:

- \( U[1, \ldots, n] \): \( E[X] = (n+1)/2 \); \( \text{var}[X] = (n^2 - 1)/12 \)
- \( B(n, p) \): \( E[X] = np \); \( \text{var}[X] = np(1-p) \).
Independence and Variance

\[ E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] \]

\( X, Y \) independent
\[ \iff Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B] \]

Then, \( f(X), g(Y) \) are independent

Also, \( E[XY] = E[X]E[Y] \) and \( \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \).

Distributions:

\( U[1, \ldots, n] : E[X] = (n + 1)/2; \text{var}[X] = (n^2 - 1)/12 \)

\( B(n, p) : E[X] = np; \text{var}[X] = np(1 - p) \).