CS70: Jean Walrand: Lecture 28.

Independence and Variance

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Independence and Variance

1. Expected value of nonnegative integer-valued RV
2. Waiting for a bus
3. A note about expectation
4. Independent random variables.
5. Variance.
6. Variance Calculations.
7. Variance Properties.

Summary of Lecture 27

Expectation; Conditional Expectation; B(n, p); G(p)

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$$E[X] = E[X|A] Pr[A] + E[X|\bar{A}] Pr[\bar{A}].$$

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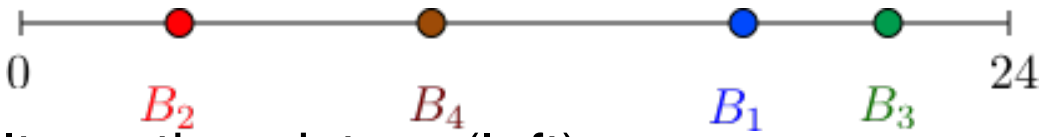

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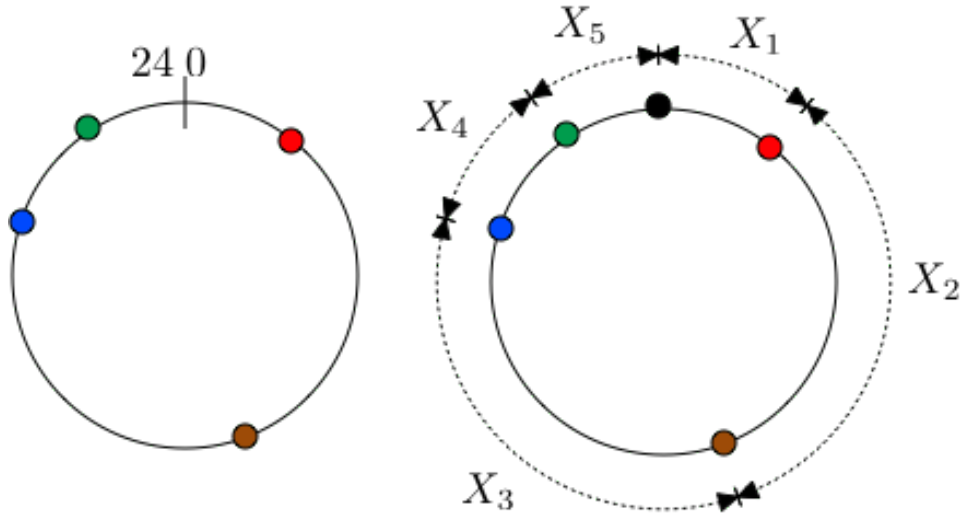

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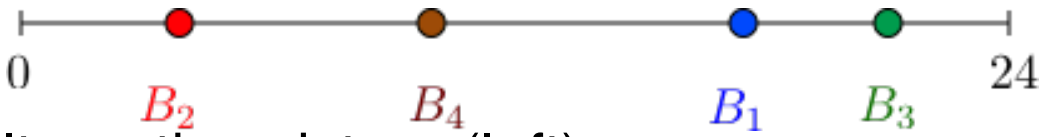

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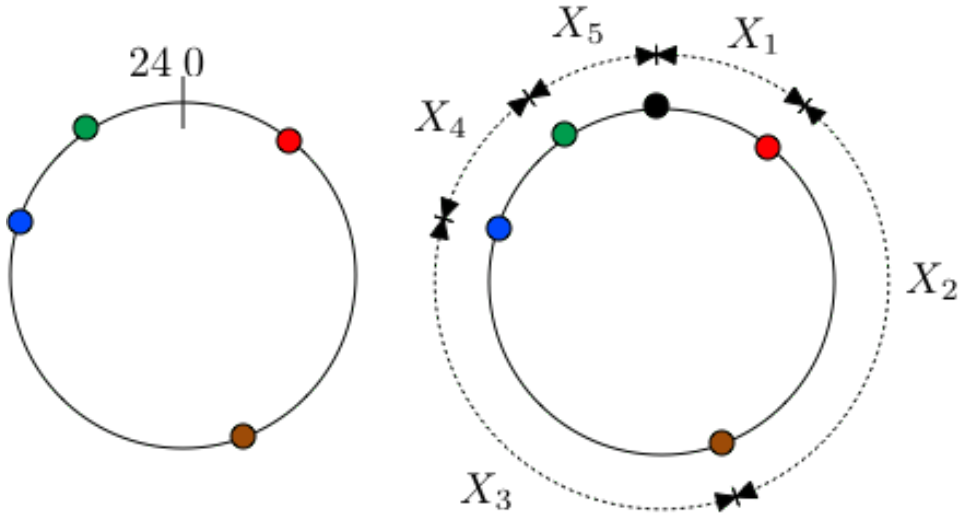

The n gaps between busses add up to 24. Thus n times the expected gap is 24 and the expected gap between busses is $24/n$.

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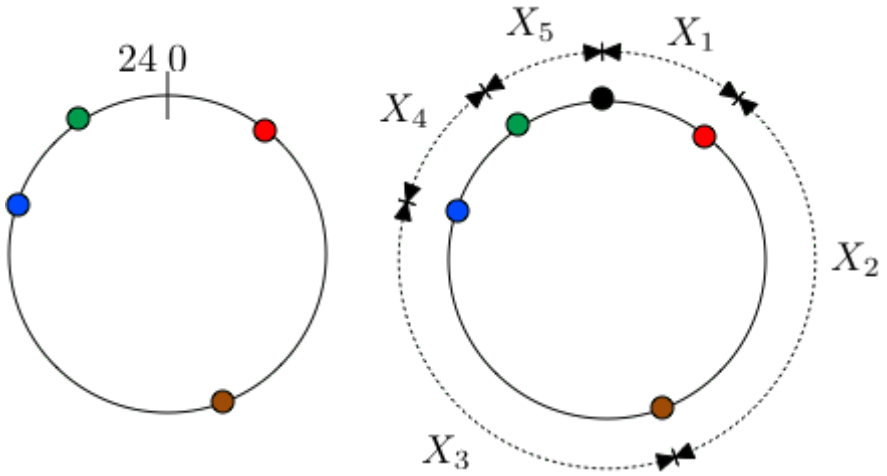

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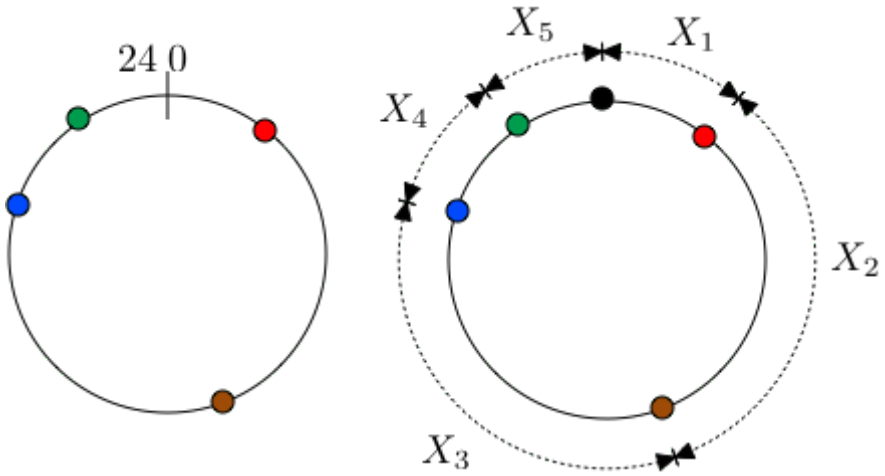

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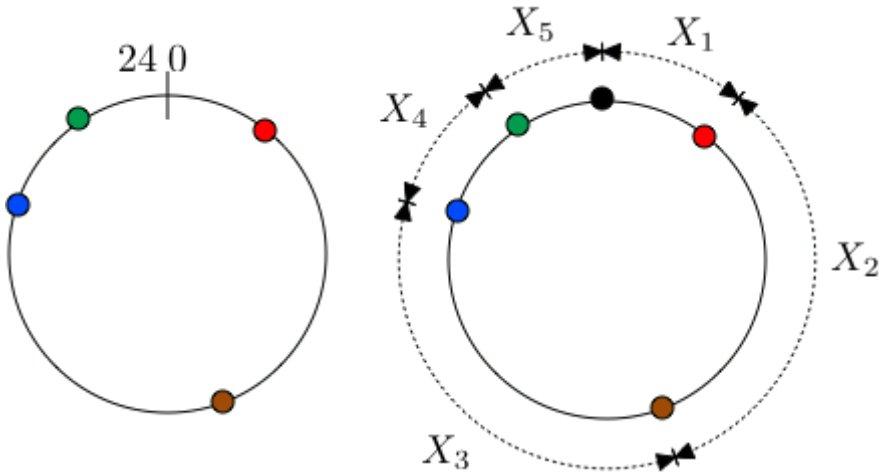

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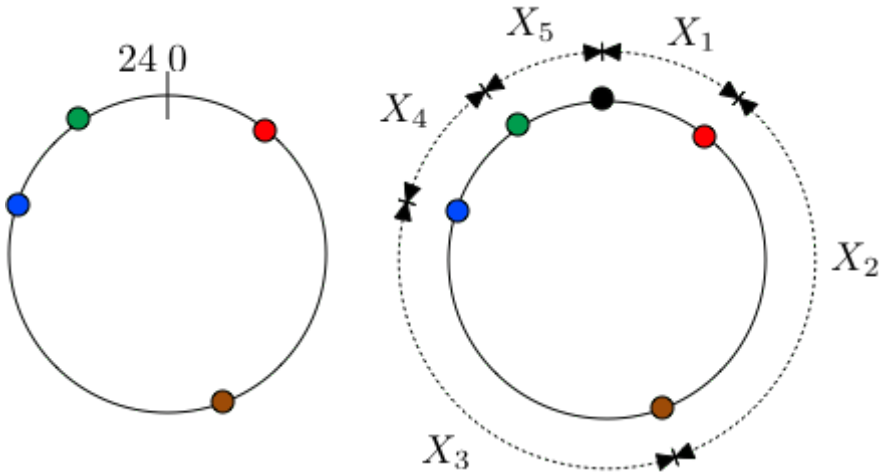

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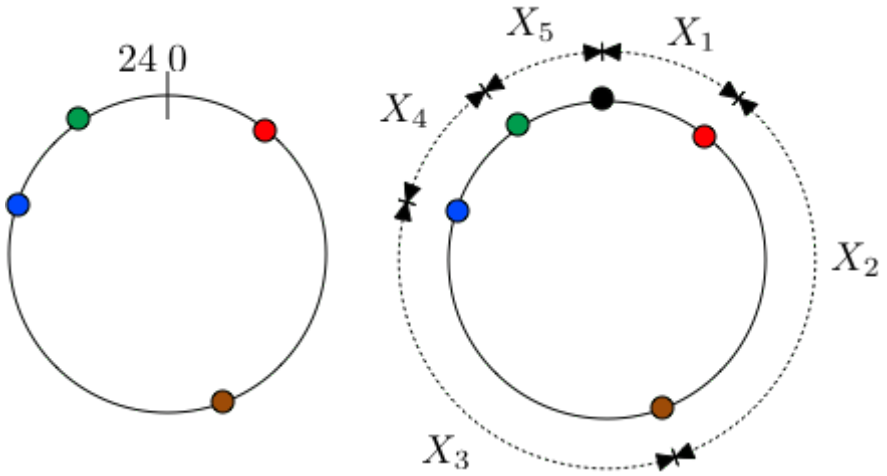

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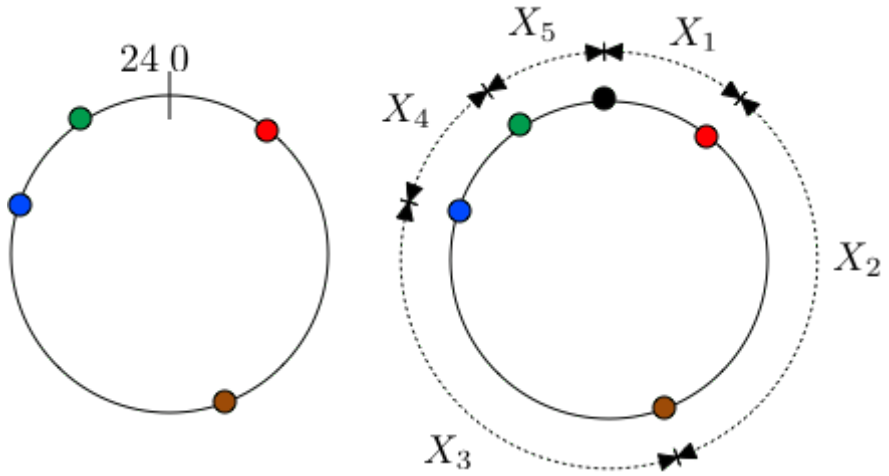

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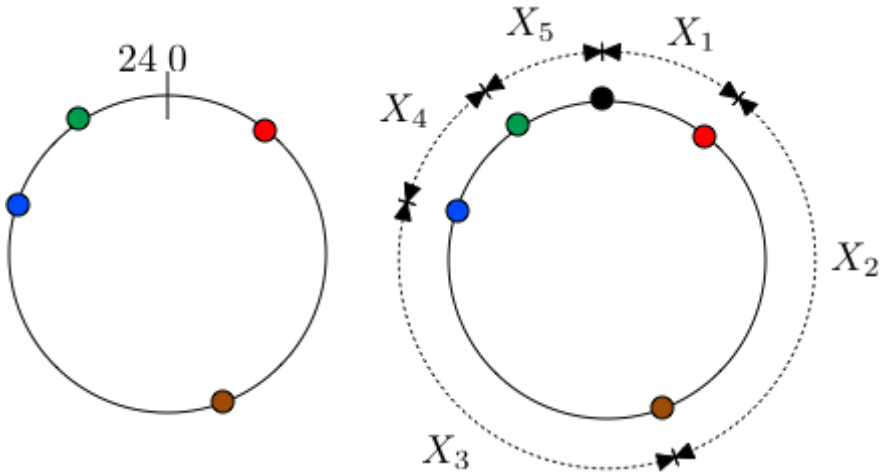

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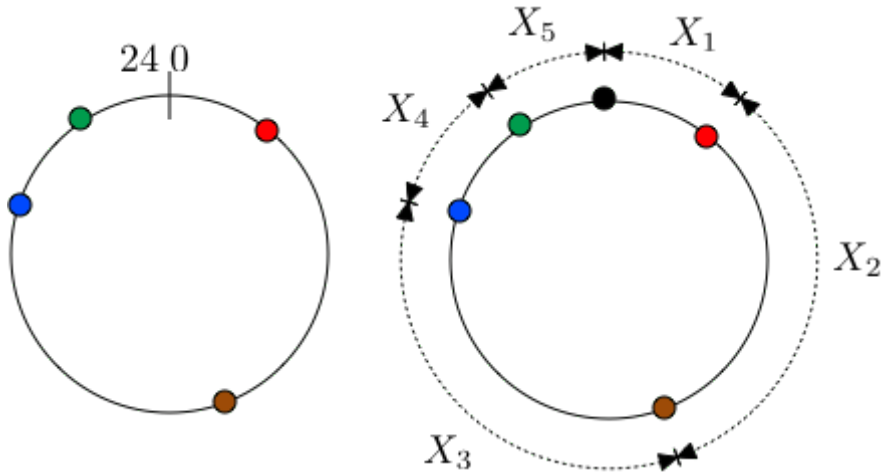

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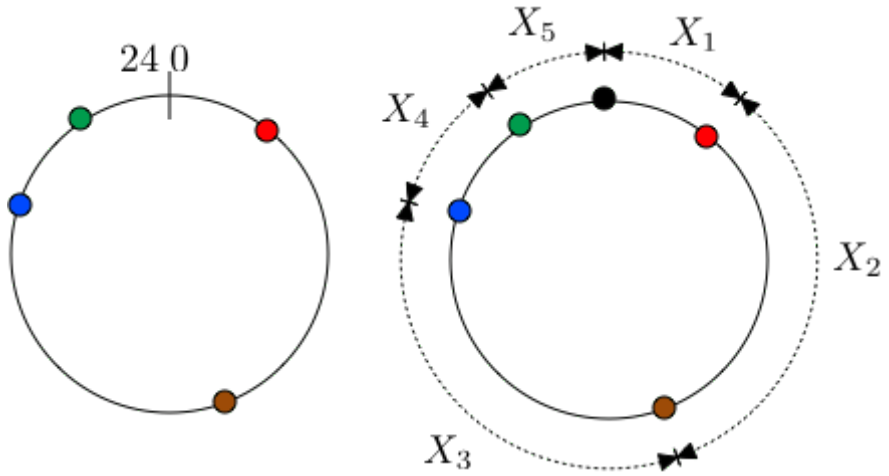

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Hence, $E[X_1] = E[X_m] = 24/5 = 24/(n+1)$ for n busses. Note the paradox: the average time between the previous and next bus is $48/(n+1) > 24/n$.

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(b) is similar. $\square$

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Mutually Independent Random Variables

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Note: If A, B, C are pairwise independent but not mutually independent, then $X = 1_A, Y = 1_B, Z = 1_C$ are pairwise independent but not mutually independent.

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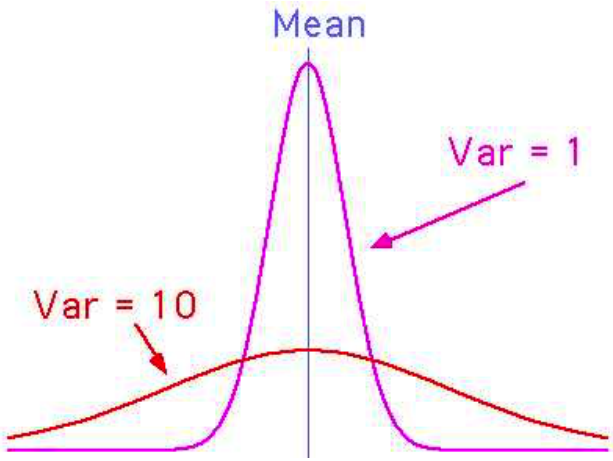
Proof:
Try it, you will like it. $\square$

Variance



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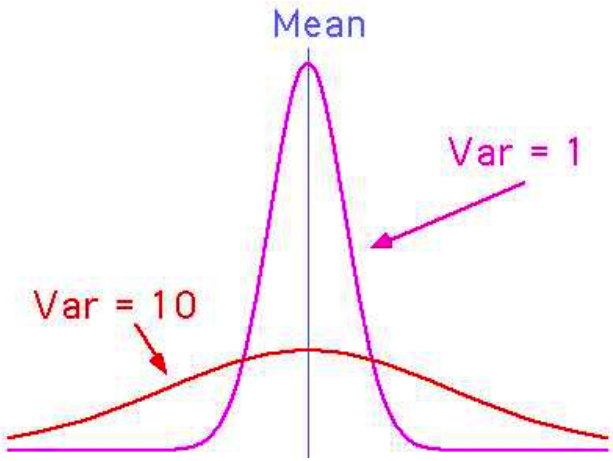

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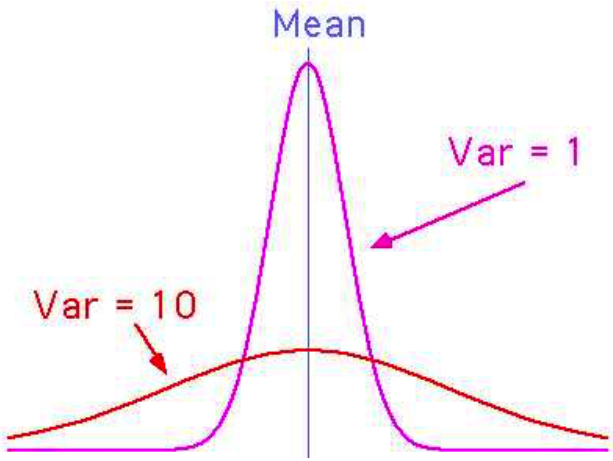

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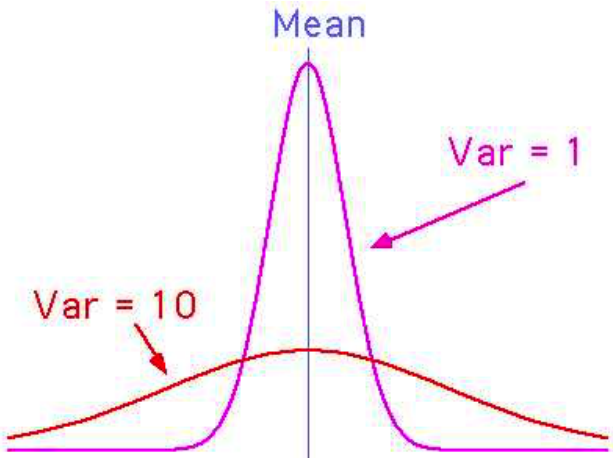

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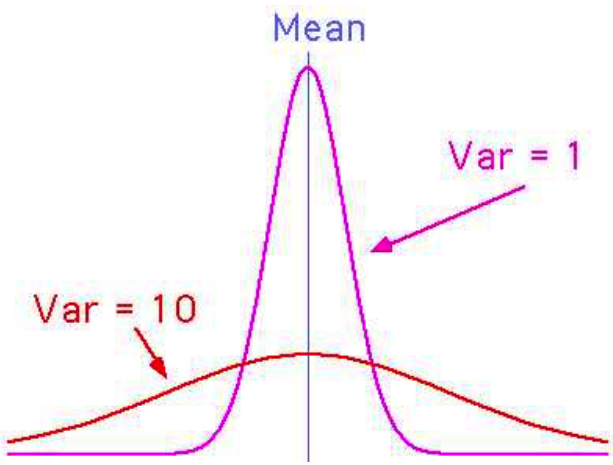

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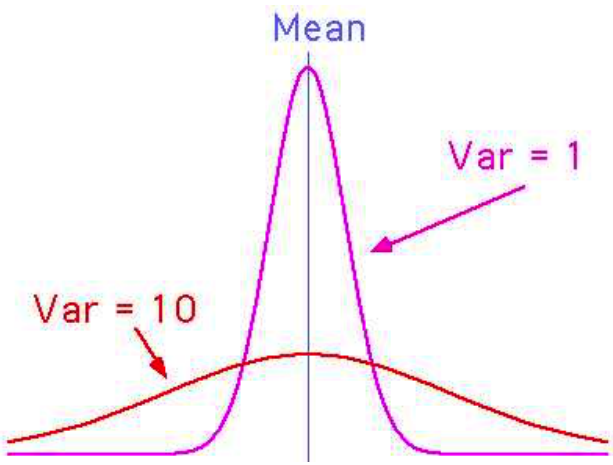

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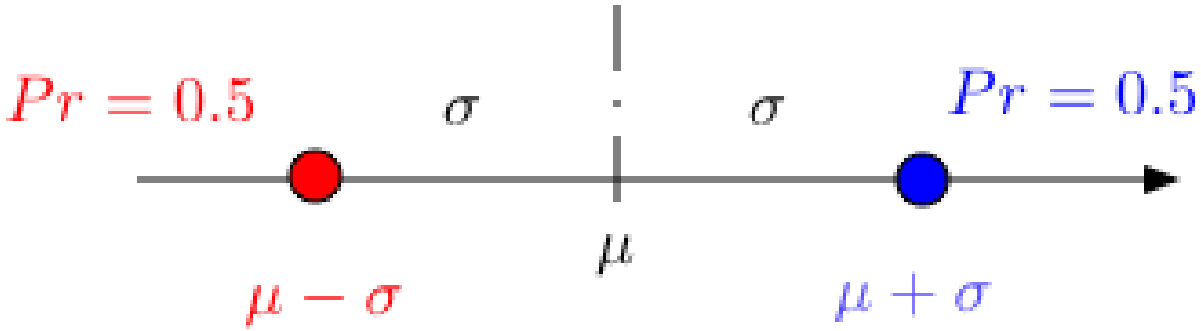

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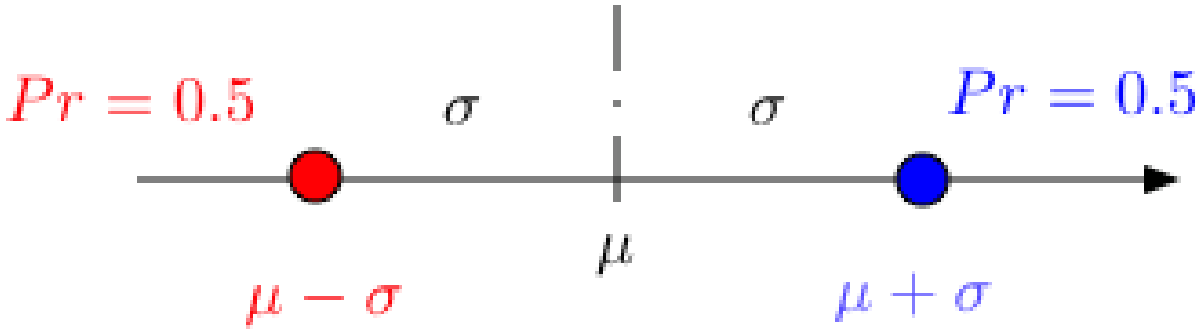

Consider the random variable X such that

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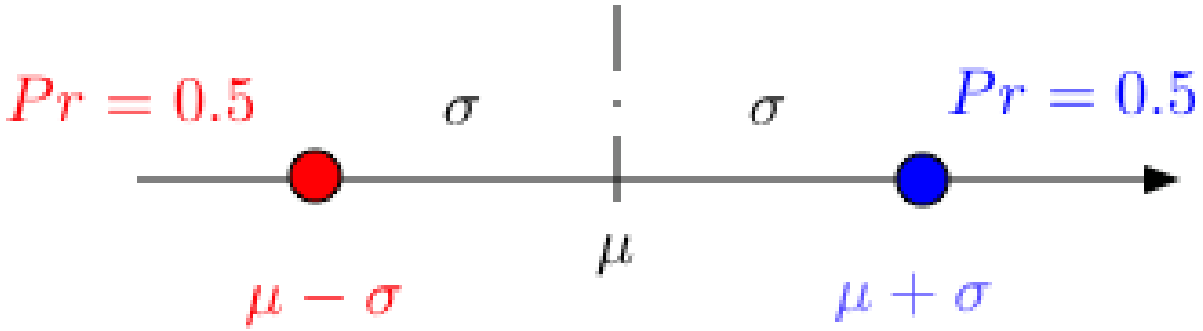

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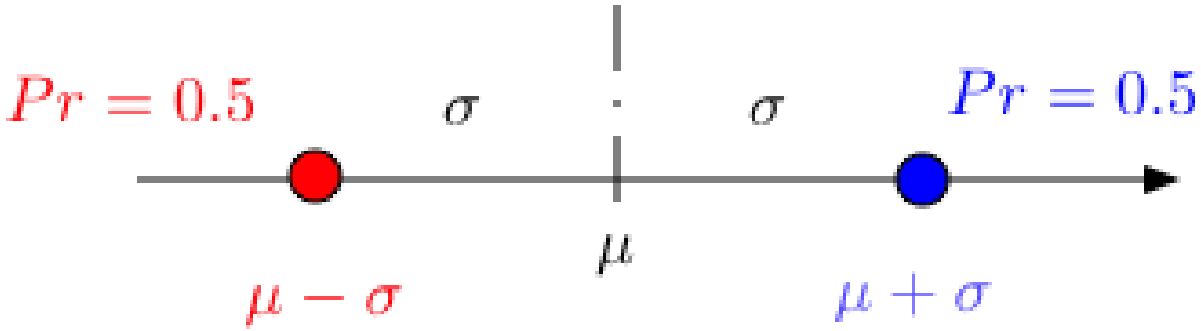

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X-E[X]|]}$?

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This gives

$$var(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$

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$Var(X) = E(X^2) - (E(X))^2 = 2 - 1 = 1.$

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Distributions:

- $U[1,\ldots,n] : E[X] = (n+1)/2; var[X] = (n^2-1)/12$
- $B(n,p) : E[X] = np; var[X] = np(1-p)$.