# CS70: Jean Walrand: Lecture 29.

Variance, Geometric, time to collect coupons; Poisson

1. Variance
2. Variance of Uniform and Geometric
3. Properties of Variance
4. Variance of $B(n, p)$
5. Coupon Collector
6. Poisson Distribution

# Review of Lecture 28

Independence and Variance

- $E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z) Pr[X = x, Y = y, Z = z]$
- $X, Y$ independent
  $\Leftrightarrow Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B]$
- Then, $f(X), g(Y)$ are independent
- Also, $E[XY] = E[X]E[Y]$
- Variance:
  $var[X] = \sigma^2(X) := E[(X - E[X])^2] = E[X^2] - E[X]^2$.

# Example

Consider $X$ with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then

$$\begin{aligned} E[X] &= -1 \times 0.99 + 99 \times 0.01 = 0. \\ E[X^2] &= 1 \times 0.99 + (99)^2 \times 0.01 \approx 100. \\ Var(X) &\approx 100 \implies \sigma(X) \approx 10. \end{aligned}$$

Also,

$$E(|X|) = 1 \times 0.99 + 99 \times 0.01 = 1.98.$$

Thus, $\sigma(X) \neq E[|X - E[X]|]$!

Exercise: How big can you make $\frac{\sigma(X)}{E[|X-E[X]|]}$?

# Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i \\ &= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}. \end{aligned}$$

Also,

$$\begin{aligned} E[X^2] &= \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 \\ &= \frac{1 + 3n + 2n^2}{6}, \text{ as you can verify.} \end{aligned}$$

This gives

$$var(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$

# Fixed points.

Number of fixed points in a random permutation of $n$ items.
$X =$ Number of student that get assignment back.

$$X = X_1 + X_2 \cdots + X_n$$

where $X_i$ is indicator variable for $i$th student getting assignment back.

$$\begin{aligned} E[X^2] &= \sum_i E[X_i^2] + \sum_{i \neq j} E[X_i X_j]. \\ &= n \times \frac{1}{n} + (n)(n-1) \times \frac{1}{n(n-1)} = 1 + 1 = 2. \end{aligned}$$

$E[X_i^2] = E[X_i] = 1/n$

$E[X_i X_j] = 1 \times Pr[X_i = 1 \text{ and } X_j = 1] + 0 \times Pr[\text{“anything else”}]$
$= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}$

$var[X] = E[X^2] - E[X]^2 = 2 - 1 = 1.$ Slide animation by Satish Rao

# Variance: binomial.

$$
\begin{aligned}
E[X^2] &= \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1-p)^{n-i}. \\
&= \text{Really???!!\#\#...}
\end{aligned}
$$

Too hard!

Ok.. fine.
Let's do something else.
Maybe not much easier...but there is a payoff.

# Properties of variance.

1. $var[cX] = c^2 var[X]$, where c is a constant.
   Scales by $c^2$.
2. $var[X+c] = var[X]$, where c is a constant.
   Shifts center.

**Proof:**

$$
\begin{aligned}
var[cX] &= E[(cX)^2] - (E[cX])^2 \\
&= c^2 E[X^2] - c^2 E[X]^2 = c^2(E[X^2] - E[X]^2) \\
&= c^2 var[X] \\
var[X+c] &= E[(X+c-E[X+c])^2] \\
&= E[(X+c-E[X]-c)^2] \\
&= E[(X-E[X])^2] = var[X]
\end{aligned}
$$

$\square$

# Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$var[X+Y] = var[X] + var[Y].$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = 0$ and $E[Y] = 0$.

Then, by independence,

$$E[XY] = E[X]E[Y] = 0.$$

Hence,

$$\begin{aligned} var[X+Y] &= E[(X+Y)^2] = E[X^2 + 2XY + Y^2] \\ &= E[X^2] + 2E[XY] + E[Y^2] = E[X^2] + E[Y^2] \\ &= var[X] + var[Y]. \end{aligned}$$

# Variance of Binomial Distribution.

Flip coin with heads probability $p$.
$X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_i^2] = E[X_i] = p$.
$var[X_i] = p - E[X]^2 = p - p^2 = p(1-p)$.

Note: $p = 0 \implies var[X_i] = 0$. Also, $p = 1 \implies var[X_i] = 0$
Now,

$$X = X_1 + X_2 + \ldots X_n.$$

$X_i$ and $X_j$ are independent. Hence,

$$var[X] = var[X_1 + \cdots X_n] = n \times var[X_1] = np(1-p).$$

# Variance of geometric distribution.

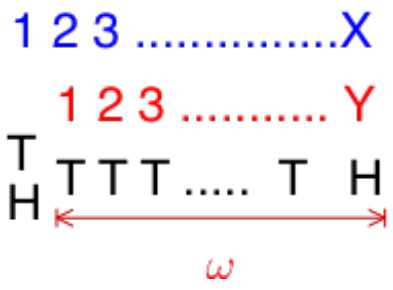
$X$ is a geometrically distributed RV with parameter $p$.
Thus, $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

$$
\begin{aligned}
E[X^2] &= p + 4p(1-p) + 9p(1-p)^2 + \ldots \\
-(1-p)E[X^2] &= \quad -[p(1-p) + 4p(1-p)^2 + \ldots] \\
pE[X^2] &= p + 3p(1-p) + 5p(1-p)^2 + \ldots \\
&= 2(p + 2p(1-p) + 3p(1-p)^2 + ..) \quad E[X]! \\
&\quad -(p + p(1-p) + p(1-p)^2 + \ldots) \quad \text{Distribution.} \\
pE[X^2] &= 2E[X] - 1 \\
&= 2(\frac{1}{p}) - 1 = \frac{2-p}{p}
\end{aligned}
$$

$\implies E[X^2] = (2-p)/p^2$ and
$var[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$.
$\sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X]$ when $p$ is small(ish).

Slide animation by Satish Rao

# Variance of G(p) by renewal




Recall our renewal trip for $X = G(p)$: Let $Z = 1$ if the first coin flip is $T$ and $Z = 0$, otherwise. Thus, $Pr[Z = 1] = q = 1 - p$. Then $Z^2 = Z$ and

$$X = 1 + ZY$$

where $Y, Z$ are independent and $X, Y$ are identically distributed.

Hence,

$$\begin{aligned} E[X^2] &= E[(1+ZY)^2] = E[1 + 2ZY + Z^2 Y^2] \\ &= 1 + 2qE[Y] + qE[Y^2] = 1 + 2q/p + qE[X^2]. \end{aligned}$$

Thus, $pE[X^2] = 1 + 2q/p = (p + 2q)/p = (2 - p)/p$, so that $E[X^2] = (2 - p)/p^2$. Hence,

$$var(X) = E[X^2] - E[X]^2 = E[X^2] - 1/p^2 = (1 - p)/p^2.$$

# Coupon Collectors Problem.

**Experiment:** Get coupons at random from $n$ until collect all $n$ coupons.
**Outcomes:** $\{123145..., 56765...\}$
**Random Variable:** $X$ - length of outcome.

Before: $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$.

Today: $E[X]$?

# Time to collect coupons

$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”}|\text{“got } \cancel{\text{milk}} \text{ first coupon”}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[\text{“getting } i\text{th coupon}|\text{“got } i-1\text{rst coupons”}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

# Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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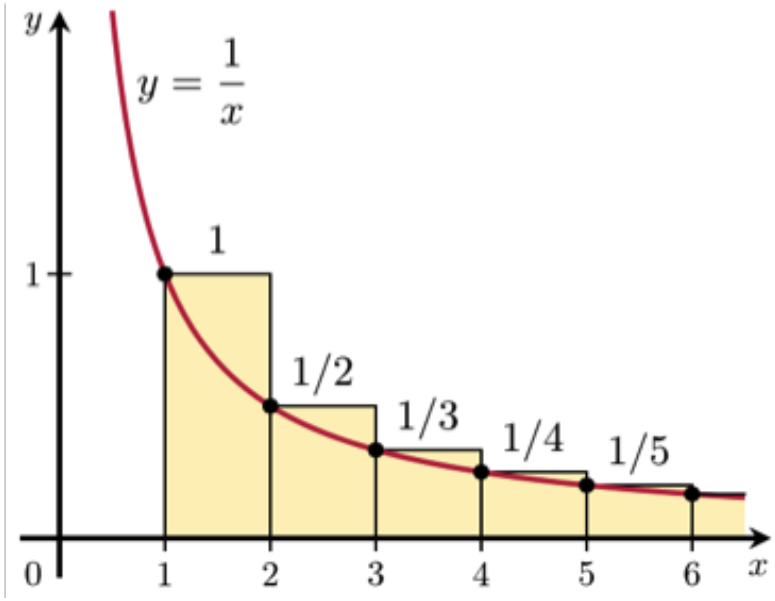
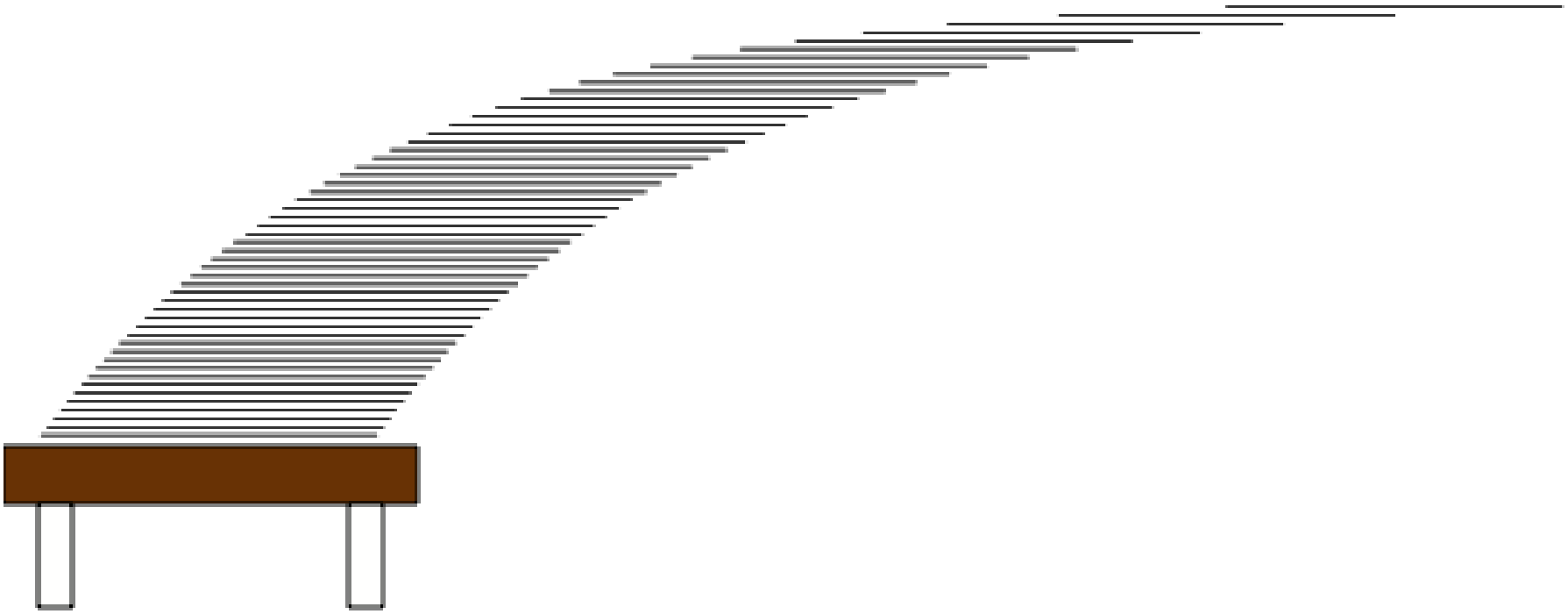



A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

# Harmonic sum: Paradox

Consider this stack of cards (no glue!):



If each card has length 2, the stack can extend $H(n)$ to the right of the table. As $n$ increases, you can go as far as you want!

# Paradox

## par·a·dox

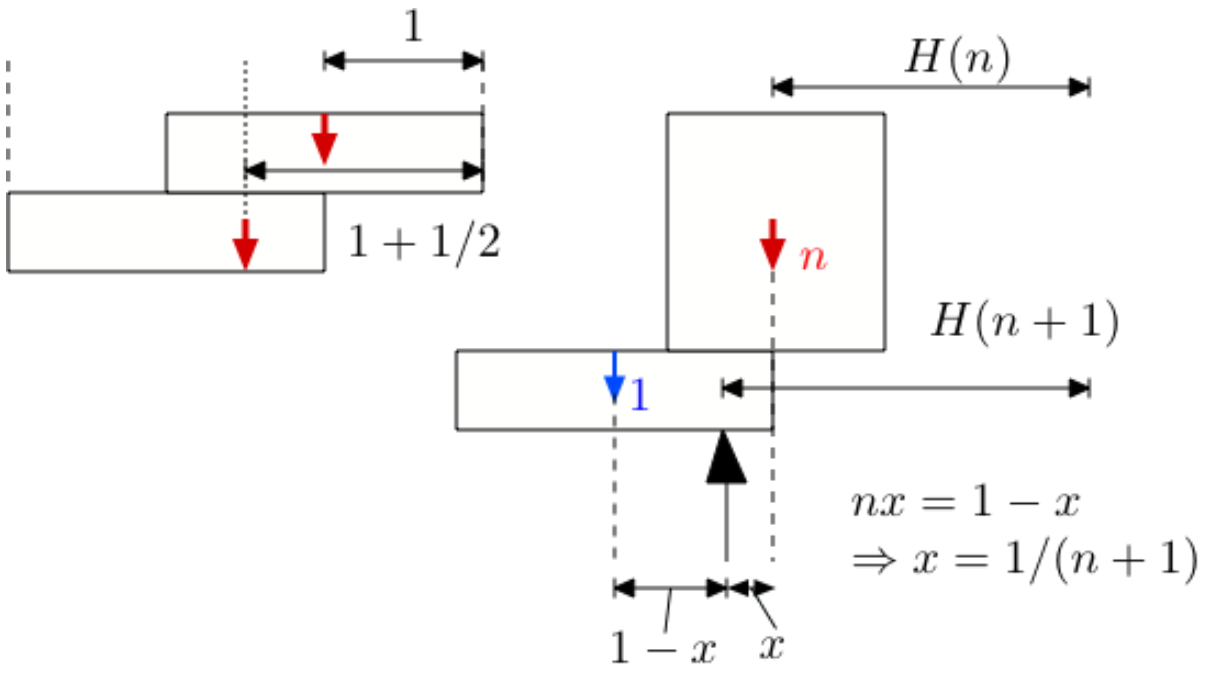
/ˈperəˌdäks/

*noun*

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  *synonyms:* contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

# Stacking



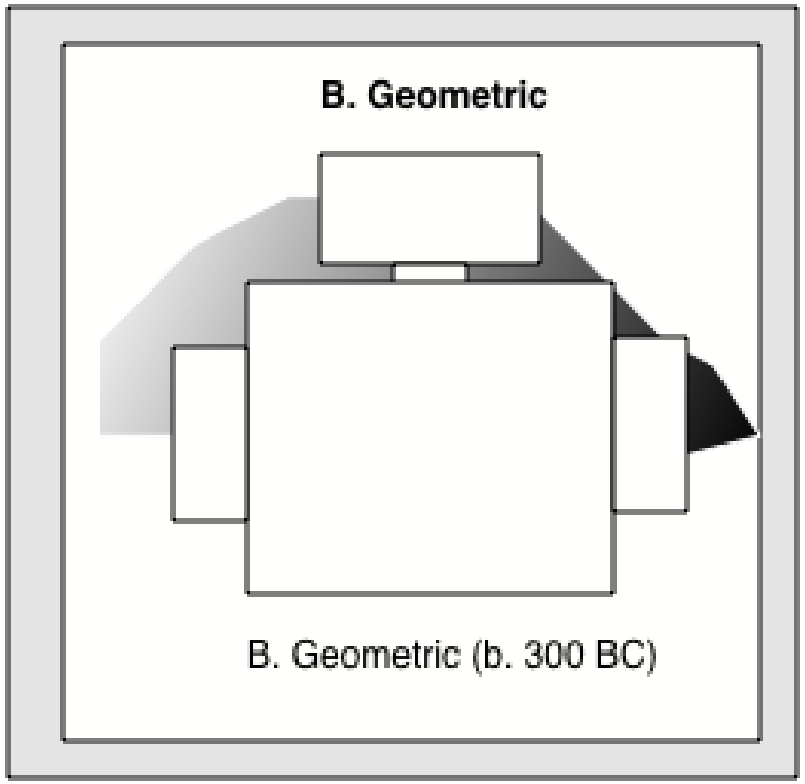

The cards have width 2. Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.

# Simeon Poisson

The Poisson distribution is named after:

# Equal Time: B. Geometric

The geometric distribution is named after:

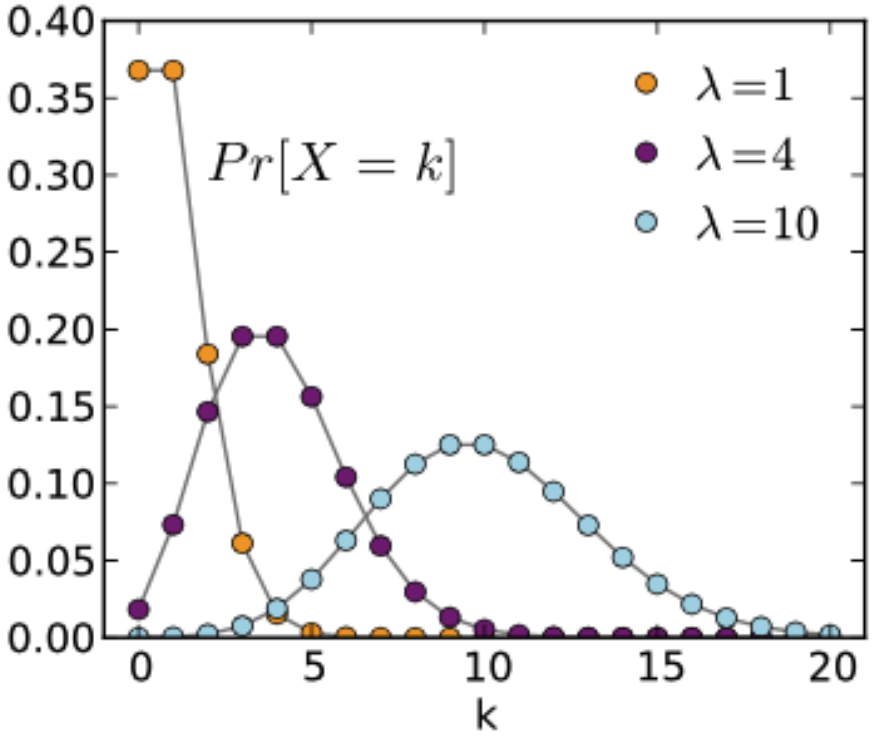



I could not find a picture of D. Binomial, sorry.

# Poisson

Experiment: flip a coin $n$ times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: $X$ - number of heads. Thus, $X = B(n, \lambda/n)$.

**Poisson Distribution** is distribution of $X$ "for large $n$."



# Poisson

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Random Variable: $X$ - number of heads. Thus, $X = B(n, \lambda/n)$.
**Poisson Distribution** is distribution of $X$ "for large $n$."

$$\begin{aligned}
Pr[X = m] &= \binom{n}{m} p^m (1-p)^{n-m}, \text{ with } p = \lambda/n \\
&= \frac{n(n-1)\cdots(n-m+1)}{m!} \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx \frac{n(n-1)\cdots(n-m+1)}{n^m} \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \\
&\approx \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n-m} \approx \frac{\lambda^m}{m!} \left(1 - \frac{\lambda}{n}\right)^{n} \approx \frac{\lambda^m}{m!} e^{-\lambda}.
\end{aligned}$$

We used $(1 - a/n)^n \approx e^{-a}$.

# Poisson Distribution: Definition and Mean

**Definition** Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

**Fact:** $E[X] = \lambda$.

**Proof:**

$$\begin{aligned} E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= e^{-\lambda} \lambda e^{\lambda} = \lambda. \end{aligned}$$

$\square$

# Poisson Distribution: Variance

Recall: $X = P(\lambda) \Leftrightarrow Pr[X = n] = \frac{\lambda^n}{n!} e^{-\lambda}, n \geq 0.$

Fact: $var[X] = \lambda$. One finds

$$\begin{aligned} E[X(X-1)] &= \sum_{n \geq 0} n(n-1) \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n \geq 2} \frac{\lambda^n}{(n-2)!} \\ &= \lambda^2 e^{-\lambda} \sum_{n \geq 2} \frac{\lambda^{n-2}}{(n-2)!} = \lambda^2 e^{-\lambda} \sum_{m \geq 0} \frac{\lambda^m}{m!} \\ &= \lambda^2. \end{aligned}$$

Hence,

$$E[X^2] - E[X] = \lambda^2.$$

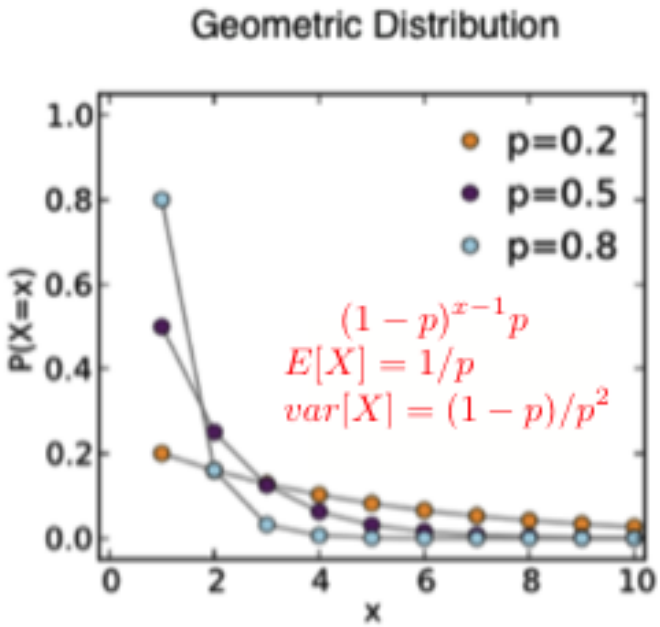
Thus,

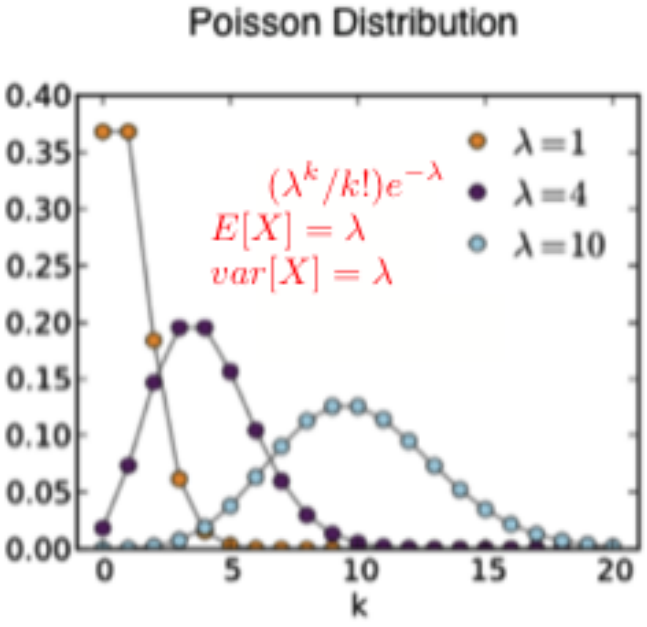
$$E[X^2] = \lambda^2 + E[X] = \lambda^2 + \lambda.$$

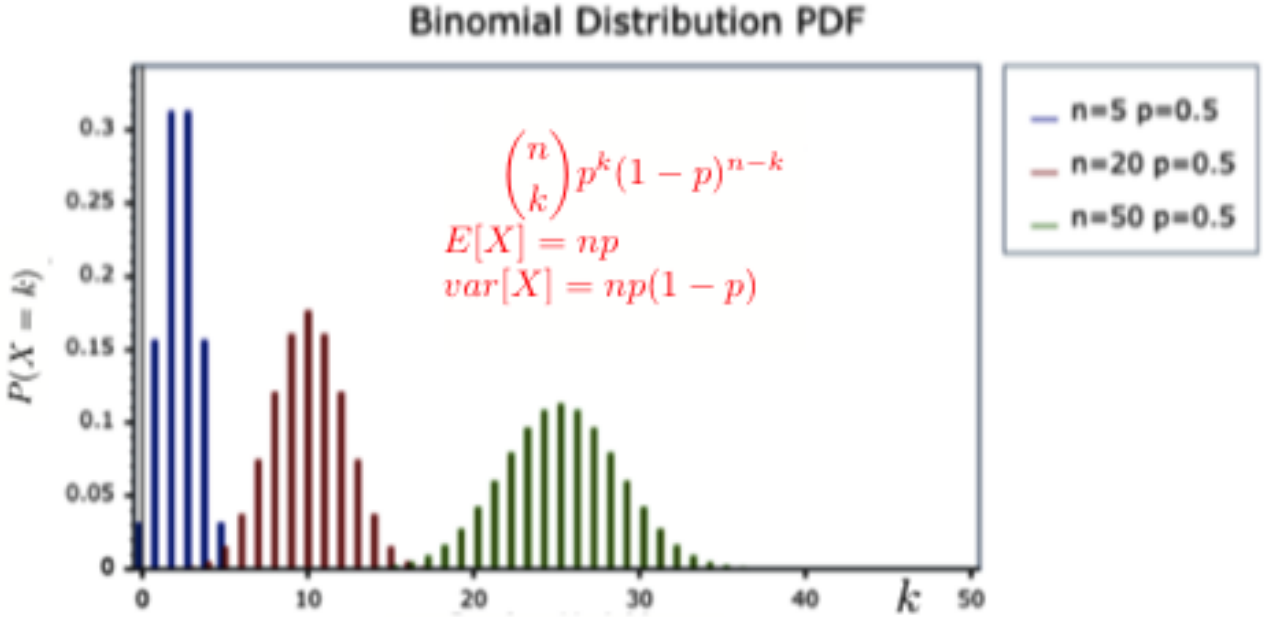
Consequently,

$$var[X] = E[X^2] - E[X]^2 = E[X^2] - \lambda^2 = \lambda.$$

# Review: Distributions



# Summary

Variance, Geometric, time to collect coupons; Poisson

- $var[a+bX] = b^2 var[X]$;
- $X, Y$ independent $\Rightarrow var[X+Y] = var[X] + var[Y]$;
- Time to collect all of $n$ coupons: $nH(n) \approx n(\ln(n) + 0.58)$;
- Intuition: Last coupon takes $1/n$ to collect!
- Remember: $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$.
- Distributions:
  - $G(p) : E[X] = 1/p, var[X] = (1-p)/p^2$;
  - $B(n,p) : E[X] = np, var[X] = np(1-p)$;
  - $P(\lambda) : E[X] = \lambda, var[X] = \lambda$