1. Variance
2. Variance of Uniform and Geometric
3. Properties of Variance
4. Variance of $B(n, p)$
5. Coupon Collector
6. Poisson Distribution
Independence and Variance

- $E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z]$
- $X, Y$ independent
  $\Leftrightarrow Pr[X \in A, Y \in B] = Pr[X \in A] Pr[Y \in B]$
- Then, $f(X), g(Y)$ are independent
- Also, $E[XY] = E[X]E[Y]$
- Variance:
  $var[X] = \sigma^2(X) := E[(X - E[X])^2] = E[X^2] - E[X]^2$. 
Example

Consider $X$ with

$$X = \begin{cases} 
-1, & \text{w. p. } 0.99 \\
99, & \text{w. p. } 0.01.
\end{cases}$$

Then

$$E[X] = -1 \times 0.99 + 99 \times 0.01 = 0.$$  
$$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100.$$  
$$\text{Var}(X) \approx 100 \implies \sigma(X) \approx 10.$$  

Also,

$$E(|X|) = 1 \times 0.99 + 99 \times 0.01 = 1.98.$$  

Thus, $\sigma(X) \neq E[|X - E[X]|]!$

Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?
Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$ 

Also,

$$E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1+3n+2n^2}{6}, \text{ as you can verify.}$$

This gives

$$\text{var}(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$
Fixed points.

Number of fixed points in a random permutation of $n$ items. $X = \text{Number of student that get assignment back.}$

$$X = X_1 + X_2 \ldots + X_n$$

where $X_i$ is indicator variable for $i$th student getting assignment back.

$$E[X^2] = \sum_i E[X_i^2] + \sum_{i \neq j} E[X_i X_j].$$

$$= n \times \frac{1}{n} + (n)(n-1) \times \frac{1}{n(n-1)} = 1 + 1 = 2.$$

$E[X_i^2] = E[X_i] = 1/n$

$E[X_i X_j] = 1 \times Pr[X_i = 1 \text{ and } X_j = 1] + 0 \times Pr[\text{"anything else"'}]$

$$= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}$$

$var[X] = E[X^2] - E[X]^2 = 2 - 1 = 1.$
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

=  Really??!!#... 

Too hard!

Ok.. fine.

Let’s do something else.

Maybe not much easier...but there is a payoff.
Properties of variance.

1. $\text{var}[cX] = c^2 \text{var}[X]$, where $c$ is a constant. Scales by $c^2$.

2. $\text{var}[X + c] = \text{var}[X]$, where $c$ is a constant. Shifts center.

Proof:

$$
\begin{align*}
\text{var}[cX] &= E[(cX)^2] - (E[cX])^2 \\
 &= c^2 E[X^2] - c^2 E[X]^2 = c^2 (E[X^2] - E[X]^2) \\
 &= c^2 \text{var}[X] \\
\text{var}[X + c] &= E[(X + c - E[X + c])^2] \\
 &= E[(X + c - E[X] - c)^2] \\
 &= E[(X - E[X])^2] = \text{var}[X]
\end{align*}
$$
Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{var}[X + Y] = \text{var}[X] + \text{var}[Y].$$

**Proof:**
Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E[X] = 0$ and $E[Y] = 0$.

Then, by independence,


Hence,

$$\text{var}[X + Y] = E[(X + Y)^2] = E[X^2 + 2XY + Y^2]$$
$$= \text{var}[X] + \text{var}[Y].$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$


$\text{var}[X_i] = p - E[X]^2 = p - p^2 = p(1 - p).$

Note: $p = 0 \implies \text{var}[X_i] = 0.$ Also, $p = 1 \implies \text{var}[X_i] = 0.$

Now,

$$X = X_1 + X_2 + \ldots X_n.$$ 

$X_i$ and $X_j$ are independent. Hence,

$$\text{var}[X] = \text{var}[X_1 + \cdots X_n] = n \times \text{var}[X_1] = np(1 - p).$$
Variance of geometric distribution.

\( X \) is a geometrically distributed RV with parameter \( p \).

Thus, \( \Pr[X = n] = (1 - p)^{n-1}p \) for \( n \geq 1 \). Recall \( E[X] = 1/p \).

\[
E[X^2] = p + 4p(1 - p) + 9p(1 - p)^2 + \ldots
\]

\[
-(1 - p)E[X^2] = -[p(1 - p) + 4p(1 - p)^2 + \ldots]
\]

\[
pE[X^2] = p + 3p(1 - p) + 5p(1 - p)^2 + \ldots
\]

\[
= 2(p + 2p(1 - p) + 3p(1 - p)^2 + \ldots)
\]

\[
- (p + p(1 - p) + p(1 - p)^2 + \ldots)
\]

\[= 2E[X] - 1\]

\[= 2\left(\frac{1}{p}\right) - 1 = \frac{2 - p}{p}\]

\[\implies E[X^2] = \frac{(2 - p)}{p^2}\]

\text{var}[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.

\( \sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X] \) when \( p \) is small(ish).
Variance of $G(p)$ by renewal

Recall our renewal trip for $X = G(p)$: Let $Z = 1$ if the first coin flip is $T$ and $Z = 0$, otherwise. Thus, $Pr[Z = 1] = q = 1 - p$. Then $Z^2 = Z$ and

$$X = 1 + ZY$$

where $Y, Z$ are independent and $X, Y$ are identically distributed. Hence,

$$E[X^2] = E[(1 + ZY)^2] = E[1 + 2ZY + Z^2 Y^2]$$

$$= 1 + 2qE[Y] + qE[Y^2] = 1 + 2q/p + qE[X^2].$$

Thus, $pE[X^2] = 1 + 2q/p = (p + 2q)/p = (2 - p)/p$, so that $E[X^2] = (2 - p)/p^2$. Hence,

Experiment: Get coupons at random from $n$ until collect all $n$ coupons.
Outcomes: $\{123145..., 56765...\}$
Random Variable: $X$ - length of outcome.
Before: $Pr[X \geq n\ln 2n] \leq \frac{1}{2}$.
Today: $E[X]$?
Time to collect coupons

- $X$: time to get $n$ coupons.
- $X_1$: time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.
- $X_2$: time to get second coupon after getting first.

$Pr["get second coupon"]|"got milk — first coupon"] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr["getting \ ith \ coupon"]|"got \ i-1st \ coupons"] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}$, $i = 1, 2, \ldots, n$.

$E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$

$= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(ln\ n + \gamma)$
Review: Harmonic sum

\[ H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_{1}^{n} \frac{1}{x} \, dx = \ln(n). \]

A good approximation is

\[ H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant)}. \]
Harmonic sum: Paradox

Consider this stack of cards (no glue!):  

If each card has length 2, the stack can extend $H(n)$ to the right of the table. As $n$ increases, you can go as far as you want!
**paradox**

/ˈpərəˌdæks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.

"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"

  *synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More*

- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"
The cards have width 2. Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.
Simeon Poisson

The Poisson distribution is named after:
Equal Time: B. Geometric

The geometric distribution is named after:

I could not find a picture of D. Binomial, sorry.
Experiment: flip a coin \( n \) times. The coin is such that \( Pr[H] = \frac{\lambda}{n} \).

Random Variable: \( X \) - number of heads. Thus, \( X = B(n, \frac{\lambda}{n}) \).

**Poisson Distribution** is distribution of \( X \) “for large \( n \).”
Poisson

Experiment: flip a coin \( n \) times. The coin is such that \( Pr[H] = \lambda / n \).
Random Variable: \( X \) - number of heads. Thus, \( X = B(n, \lambda / n) \).
**Poisson Distribution** is distribution of \( X \) “for large \( n \).”

\[
Pr[X = m] = \binom{n}{m} p^m (1 - p)^{n-m}, \text{ with } p = \lambda / n
\]

\[
= \frac{n(n-1) \cdots (n-m+1)}{m!} \left( \frac{\lambda}{n} \right)^m \left( 1 - \frac{\lambda}{n} \right)^{n-m}
\]

\[
\approx \frac{n(n-1) \cdots (n-m+1)}{n^m} \frac{\lambda^m}{m!} \left( 1 - \frac{\lambda}{n} \right)^{n-m}
\]

\[
\approx \frac{\lambda^m}{m!} \left( 1 - \frac{\lambda}{n} \right)^{n-m} \approx \frac{\lambda^m}{m!} \left( 1 - \frac{\lambda}{n} \right)^n \approx \frac{\lambda^m}{m!} e^{-\lambda}.
\]

We used \((1 - a/n)^n \approx e^{-a}\).
Poisson Distribution: Definition and Mean

**Definition** Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, \ m \geq 0.$$  

**Fact:** $E[X] = \lambda$.

**Proof:**

$$E[X] = \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!}$$

$$= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}$$

$$= e^{-\lambda} \lambda e^\lambda = \lambda.$$
Poisson Distribution: Variance

Recall: \( X = P(\lambda) \Leftrightarrow \Pr[X = n] = \frac{\lambda^n}{n!} e^{-\lambda}, n \geq 0. \)

Fact: \( \text{var}[X] = \lambda. \) One finds

\[
E[X(X - 1)] = \sum_{n \geq 0} n(n-1) \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n \geq 2} \frac{\lambda^n}{(n-2)!}
\]

\[
= \lambda^2 e^{-\lambda} \sum_{n \geq 2} \frac{\lambda^{n-2}}{(n-2)!} = \lambda^2 e^{-\lambda} \sum_{m \geq 0} \frac{\lambda^m}{m!}
\]

\[
= \lambda^2.
\]

Hence,

\[
E[X^2] - E[X] = \lambda^2.
\]

Thus,

\[
E[X^2] = \lambda^2 + E[X] = \lambda^2 + \lambda.
\]

Consequently,

\[
\text{var}[X] = E[X^2] - E[X]^2 = E[X^2] - \lambda^2 = \lambda.
\]
Review: Distributions

Geometric Distribution

\[ P(X = x) = (1 - p)^{x-1}p \]

\[ E[X] = \frac{1}{p} \]

\[ var[X] = \frac{(1 - p)}{p^2} \]

Poisson Distribution

\[ P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda} \]

\[ E[X] = \lambda \]

\[ var[X] = \lambda \]

Binomial Distribution PDF

\[ \binom{n}{k} p^k (1 - p)^{n-k} \]

\[ E[X] = np \]

\[ var[X] = np(1 - p) \]
Summary

- Variance, Geometric, time to collect coupons; Poisson

- \( \text{Var}[a + bX] = b^2 \text{Var}[X] \);
- \( X, Y \) independent \( \Rightarrow \text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] \);
- Time to collect all of \( n \) coupons: \( nH(n) \approx n(\ln(n) + 0.58) \);
- Intuition: Last coupon takes \( 1/n \) to collect!
- Remember: \( E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n] \).
- Distributions:
  - \( G(p) : E[X] = 1/p, \text{Var}[X] = (1 - p)/p^2 \);
  - \( B(n, p) : E[X] = np, \text{Var}[X] = np(1 - p) \);
  - \( P(\lambda) : E[X] = \lambda, \text{Var}[X] = \lambda \)