Variance, Geometric, time to collect coupons; Poisson
Variance, Geometric, time to collect coupons; Poisson

1. Variance
2. Variance of Uniform and Geometric
3. Properties of Variance
4. Variance of $B(n,p)$
5. Coupon Collector
6. Poisson Distribution
Independence and Variance

\[ E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z) \Pr[X=x,Y=y,Z=z] \]

\[ X, Y \text{ independent} \iff \Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \]

\[ \text{Then, } f(X), g(Y) \text{ are independent} \]

\[ E[XY] = E[X]E[Y] \]

\[ \text{Variance: } \text{var}[X] = \sigma_X^2(X) := E[(X - E[X])^2] = E[X^2] - E[X]^2. \]
Review of Lecture 28

Independence and Variance

\[ E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] \]
Independence and Variance

- \( E[g(X, Y, Z)] = \sum_{x,y,z} g(x, y, z) Pr[X = x, Y = y, Z = z] \)
- \( X, Y \) independent
  \[ \Leftrightarrow Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B] \]
Review of Lecture 28

Independence and Variance

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Independence and Variance

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- \( X, Y \) independent \( \iff \Pr[X \in A, Y \in B] = \Pr[X \in A] \Pr[Y \in B] \)
- Then, \( f(X), g(Y) \) are independent
- Also, \( E[XY] = E[X]E[Y] \)
Independence and Variance

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- Also, \( E[XY] = E[X] E[Y] \)
- Variance:
  \( \text{var}[X] = \sigma^2(X) := E[(X - E[X])^2] = E[X^2] - E[X]^2. \)
Example

Consider $X$ with

$$X = \begin{cases} -1, & \text{w. p. } 0.99 \\ 99, & \text{w. p. } 0.01. \end{cases}$$

Then $E[X] = -1 \times 0.99 + 99 \times 0.01 = 0$.

$E[X^2] = 1 \times 0.99 + (99)^2 \times 0.01 \approx 100$. 

$\text{Var}(X) \approx 100 = \Rightarrow \sigma(X) \approx 10$.

Also, $E(|X|) = -1 \times 0.99 + 99 \times 0.01 = 1.98$.

Thus, $\sigma(X) \neq E[|X - E[X]|]$.

Exercise: How big can you make $\sigma(X)$ $E[|X - E[X]|]$?
Example

Consider $X$ with

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**Example**

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Thus, $\sigma(X) \neq E[|X - E[X]|]$!

Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i$$
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$$= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$
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\]

Also,

\[
E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2
\]

\[
= \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{1}{6} n(n+1)(2n+1) = \frac{n(n+1)(2n+1)}{6}.
\]
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Also,

$$E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1 + 3n + 2n^2}{6},$$
Assume that \( Pr[X = i] = 1/n \) for \( i \in \{1, \ldots, n\} \). Then

\[
E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.
\]

Also,

\[
E[X^2] = \sum_{i=1}^{n} i^2 Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1 + 3n + 2n^2}{6}, \text{ as you can verify.}
\]
Uniform

Assume that $Pr[X = i] = 1/n$ for $i \in \{1, \ldots, n\}$. Then

$$E[X] = \sum_{i=1}^{n} i \times Pr[X = i] = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$ 

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as you can verify.

This gives

$$\text{var}(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$
Fixed points.

Number of fixed points in a random permutation of $n$ items.
Fixed points.

Number of fixed points in a random permutation of \( n \) items. 
\( X = \) Number of student that get assignment back.
Fixed points.

Number of fixed points in a random permutation of $n$ items. $X = \text{Number of student that get assignment back.}$

$$X = X_1 + X_2 \cdots + X_n$$
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Number of fixed points in a random permutation of $n$ items. $X$ = Number of student that get assignment back.

$$X = X_1 + X_2 \cdots + X_n$$

where $X_i$ is indicator variable for $i$th student getting assignment back.
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\( E[X_i X_j] = 1 \times Pr[X_i = 1 \text{ and } X_j = 1] + 0 \times Pr["anything else"] \)

\[ = \frac{1 \times 1 \times (n-2)!}{n!} \]
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$E[X_i^2] = E[X_i] = 1/n$

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$$= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}$$

$var[X] = E[X^2] − E[X]^2$
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\[ = \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)} \]

\[ var[X] = E[X^2] - E[X]^2 = 2 - 1 \]
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\text{var}[X] = E[X^2] - E[X]^2 = 2 - 1 = 1.
\]

Slide animation by Satish Rao
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]
Variance: binomial.

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\[ = \text{Really??!!##...} \]

Too hard!
Variance: binomial.

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Ok..
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Ok.. fine.
Variance: binomial.

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Too hard!

Ok.. fine.
Let’s do something else.
Variance: binomial.

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E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}.
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= Really??!!##... 

Too hard!

Ok.. fine.

Let’s do something else.

Maybe not much easier...
Variance: binomial.

\[ E[X^2] = \sum_{i=0}^{n} i^2 \binom{n}{i} p^i (1 - p)^{n-i}. \]

= Really??!!#... 

Too hard! 

Ok.. fine. 

Let’s do something else. 

Maybe not much easier...but there is a payoff.
Properties of variance.

1. \( \text{var}[cX] = c^2 \text{var}[X] \), where \( c \) is a constant.
Properties of variance.

1. \( \text{var}[cX] = c^2 \text{var}[X] \), where \( c \) is a constant.  
   Scales by \( c^2 \).
Properties of variance.

1. $\text{var}[cX] = c^2 \text{var}[X]$, where $c$ is a constant.
   Scales by $c^2$.
2. $\text{var}[X + c] = \text{var}[X]$, where $c$ is a constant.
Properties of variance.

1. \( \text{var}[cX] = c^2 \text{var}[X] \), where \( c \) is a constant.
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Variance of sum of independent random variables

**Theorem:**
If $X$ and $Y$ are independent, then

$$\text{var}[X + Y] = \text{var}[X] + \text{var}[Y].$$
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Variance of Binomial Distribution.

Flip coin with heads probability $p$. 
Variance of Binomial Distribution.

Flip coin with heads probability $p$. 
$X$- how many heads?
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{var} \left[ X_i \right] = p - \left( p \right)^2 = p(1-p).$$

Note: $p = 0 \Rightarrow \text{var} \left[ X_i \right] = 0$. Also, $p = 1 \Rightarrow \text{var} \left[ X_i \right] = 0$.

Now, $X = X_1 + \ldots + X_n$. $X_i$ and $X_j$ are independent. Hence,

$$\text{var} \left[ X \right] = n \times \text{var} \left[ X_1 \right] = np(1-p).$$
Variance of Binomial Distribution.

Flip coin with heads probability $p$.
$X$- how many heads?

\[
X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
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\end{cases}
\]

$E[X_i^2]$
Variance of Binomial Distribution.

Flip coin with heads probability $p$.
$X$- how many heads?

$$X_i = \begin{cases} 1 & \text{if } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i^2] = E[X_i]$$
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Variance of Binomial Distribution.

Flip coin with heads probability \( p \).

\( X \)- how many heads?

\[
X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\]

\[
\text{var}[X_i] = p - E[X]^2
\]
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

$$X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
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Variance of Binomial Distribution.

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Note: $p = 0 \implies \text{var}[X_i] = 0.$
Variance of Binomial Distribution.

Flip a coin with heads probability $p$. Let $X$ be the number of heads?

$$X_i = \begin{cases} 
1 & \text{if $i$th flip is heads} \\
0 & \text{otherwise} 
\end{cases}$$

$$\text{var}[X_i] = p - E[X]^2 = p - p^2 = p(1 - p).$$ 

Note: $p = 0 \implies \text{var}[X_i] = 0$. Also, $p = 1$
Variance of Binomial Distribution.

Flip coin with heads probability $p$. $X$- how many heads?

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Now,

$$X = X_1 + X_2 + \ldots X_n.$$
Variance of Binomial Distribution.

Flip coin with heads probability \( p \).

\( X \)- how many heads?

\[
X_i = \begin{cases} 
1 & \text{if } i\text{th flip is heads} \\
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\[
\]

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$$var[X] = var[X_1 + \cdots X_n] = n \times var[X_1]$$
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Variance of geometric distribution.

$X$ is a geometrically distributed RV with parameter $p$. 

Recall $E[X] = 1/p$. 

$E[X^2] = p + 4p(1-p) + 9p(1-p)^2 + \ldots$ 

$= p + 3p(1-p) + 5p(1-p)^2 + \ldots$ 

$= 2(p + 2p(1-p) + 3p(1-p)^2 + \ldots)$ 

$\Rightarrow E[X^2] = (2 - p)/p^2$ 

and 

$\text{var}(X) = E[X^2] - [E[X]]^2 = (2 - p)/p^2 - 1/p^2 = 1 - p/p^2 \approx E[X]$ when $p$ is small(ish).
Variance of geometric distribution.

$X$ is a geometrically distributed RV with parameter $p$. Thus, $Pr[X = n] = (1 - p)^{n-1}p$ for $n \geq 1$. 

$E[X] = \frac{1}{p}$.

$E[X^2] = p + 4p(1 - p) + 9p(1 - p)^2 + ...$ 

$\frac{PE[X^2]}{p} = 2\left(p + 2p(1 - p) + 3p(1 - p)^2 + ...ight)$ 

$\sigma(X) = \sqrt{1 - \frac{p}{p^2}} \approx E[X]$ when $p$ is small(ish).
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E[X^2] = 2(p + 2p(1 - p) + 3p(1 - p)^2 + \ldots)

\Rightarrow E[X^2] = \frac{2}{p^2} - \frac{1}{p}

\Rightarrow \sigma(X) = \sqrt{\frac{1 - p}{p^2}} \approx E[X] \text{ when } p \text{ is small(ish).}
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E[X^2] = p + 4p(1 - p) + 9p(1 - p)^2 + \ldots
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$$-(1-p)E[X^2] = -[p(1-p) + 4p(1-p)^2 + \ldots]$$

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pE[X^2] = p + 3p(1 - p) + 5p(1 - p)^2 + ...
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pE[X^2] = p + 3p(1 - p) + 5p(1 - p)^2 + \ldots \\
= 2(p + 2p(1 - p) + 3p(1 - p)^2 + \ldots) - (p + p(1 - p) + p(1 - p)^2 + \ldots) \\
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pE[X^2] = 2E[X] - 1 \]
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\[
\begin{align*}
E[X^2] &= p + 4p(1 - p) + 9p(1 - p)^2 + ... \\
-(1 - p)E[X^2] &= -[p(1 - p) + 4p(1 - p)^2 + ...] \\
pE[X^2] &= p + 3p(1 - p) + 5p(1 - p)^2 + ... \\
&= 2(p + 2p(1 - p) + 3p(1 - p)^2 + ... - (p + p(1 - p) + p(1 - p)^2 + ...))
\end{align*}
\]

$\implies E[X]$!

\[
\begin{align*}
pE[X^2] &= 2E[X] - 1 \\
&= 2\left(\frac{1}{p}\right) - 1
\end{align*}
\]
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\quad = 2(p + 2p(1 - p) + 3p(1 - p)^2 + ...) \\n\quad = (p + p(1 - p) + p(1 - p)^2 + ...) \\n\quad = 2E[X] - 1 \\n\quad = 2\left(\frac{1}{p}\right) - 1 = \frac{2 - p}{p}
\]
Variance of geometric distribution.

$X$ is a geometrically distributed RV with parameter $p$. Thus, $Pr[X = n] = (1 - p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

\[
E[X] = \frac{1}{p}
\]

\[
E[X^2] = p + 4p(1 - p) + 9p(1 - p)^2 + ... \\
-(1 - p)E[X^2] = -[p(1 - p) + 4p(1 - p)^2 + ...] \\
pE[X^2] = p + 3p(1 - p) + 5p(1 - p)^2 + ...
\]

\[
pE[X^2] = 2(p + 2p(1 - p) + 3p(1 - p)^2 + ..) \Rightarrow E[X]!
\]

\[
h = p + p(1 - p) + p(1 - p)^2 + ...
\]

\[
h = (1 - p)E[X^2] \\
pE[X^2] = 2E[X] - 1
\]

\[
pE[X^2] = 2\left(\frac{1}{p}\right) - 1 = \frac{2 - p}{p}
\]

\[
\Rightarrow E[X^2] = \frac{2 - p}{p^2}
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\]
\[ = 2(p + 2p(1 - p) + 3p(1 - p)^2 + ..) \quad E[X]^\dagger \]
\[ -(p + p(1 - p) + p(1 - p)^2 + ...) \quad \text{Distribution.} \]
\[ pE[X^2] = 2E[X] - 1 \]
\[ = 2\left(\frac{1}{p}\right) - 1 = \frac{2 - p}{p} \]

$\implies E[X^2] = (2 - p)/p^2$ and $\text{var}[X] = E[X^2] - E[X]^2$
Variance of geometric distribution.

$X$ is a geometrically distributed RV with parameter $p$. Thus, $Pr[X = n] = (1 - p)^{n-1}p$ for $n \geq 1$. Recall $E[X] = 1/p$.

\[
\begin{align*}
E[X^2] &= p + 4p(1 - p) + 9p(1 - p)^2 + ... \\
-(1 - p)E[X^2] &= -[p(1 - p) + 4p(1 - p)^2 + ...] \\
pE[X^2] &= p + 3p(1 - p) + 5p(1 - p)^2 + ... \\
&= 2(p + 2p(1 - p) + 3p(1 - p)^2 + ..) E[X]! \\
P &= -(p + p(1 - p) + p(1 - p)^2 + ...) \text{ Distribution.} \\
pE[X^2] &= 2E[X] - 1 \\
&= 2\left(\frac{1}{p}\right) - 1 = \frac{2 - p}{p}
\end{align*}
\]

$\implies E[X^2] = (2 - p)/p^2$ and

$\text{var}[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2}$
Variance of geometric distribution.

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\[= 2(p + 2p(1 - p) + 3p(1 - p)^2 + ..) \quad E[X]!
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\[-(p + p(1 - p) + p(1 - p)^2 + ...) \quad \text{Distribution.}
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\[pE[X^2] = 2E[X] - 1
\]
\[= 2\left(\frac{1}{p}\right) - 1 = \frac{2 - p}{p}
\]

\[\Rightarrow E[X^2] = (2 - p)/p^2 \quad \text{and}
\]
\[\text{var}[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}
\]
\[\sigma(X) = \frac{\sqrt{1-p}}{p}
\]
Variance of geometric distribution.

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\[
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\[
\implies E[X^2] = (2 - p)/p^2 \quad \text{and}
\]

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\text{var}[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.
\]

\[
\sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X] \quad \text{when} \ p \text{ is small(ish).}
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\[\implies E[X^2] = (2 - p)/p^2 \text{ and} \]
\[\text{var}[X] = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}. \]
\[\sigma(X) = \frac{\sqrt{1-p}}{p} \approx E[X] \text{ when } p \text{ is small(ish)}. \]
Variance of $G(p)$ by renewal

Recall our renewal trip for $X = G(p)$: Let $Z = 1$ if the first coin flip is $T$ and $Z = 0$, otherwise. Thus, $\Pr[Z = 1] = q = 1 - p$.

Then $Z^2 = Z$ and $X = 1 + ZY$ where $Y, Z$ are independent and $X, Y$ are identically distributed.


Thus, $pE[X^2] = 1 + 2q/p = (p + 2q)/p = (2 - p)/p$, so that $E[X^2] = (2 - p)/p^2$. Hence, $\text{var}(X) = E[X^2] - E[X]^2 = \frac{1 - p}{p^2}$. 

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Hence,

$$E[X^2] = E[(1 + ZY)^2] = E[1 + 2ZY + Z^2Y^2]$$
Variance of G(p) by renewal

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Hence,

\[
E[X^2] = E[(1 + ZY)^2] = E[1 + 2ZY + Z^2Y^2]
\]

\[
= 1 + 2qE[Y] + qE[Y^2]
\]
Variance of $G(p)$ by renewal

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where $Y, Z$ are independent and $X, Y$ are identically distributed.

Hence,

$$E[X^2] = E[(1 + ZY)^2] = E[1 + 2ZY + Z^2Y^2]$$

$$= 1 + 2qE[Y] + qE[Y^2] = 1 + 2q/p + qE[X^2].$$

Thus, $pE[X^2] = 1 + 2q/p = (p + 2q)/p = (2 - p)/p$, so that $E[X^2] = (2 - p)/p^2$. 
Variance of $G(p)$ by renewal

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$$E[X^2] = E[(1 + ZY)^2] = E[1 + 2ZY + Z^2Y^2]$$

$$= 1 + 2qE[Y] + qE[Y^2] = 1 + 2q/p + qE[X^2].$$

Thus, $pE[X^2] = 1 + 2q/p = (p + 2q)/p = (2 - p)/p$, so that $E[X^2] = (2 - p)/p^2$. Hence,

$$var(X) = E[X^2] - E[X]^2 =$$
Variance of $G(p)$ by renewal

Recall our renewal trip for $X = G(p)$: Let $Z = 1$ if the first coin flip is $T$ and $Z = 0$, otherwise. Thus, $Pr[Z = 1] = q = 1 - p$. Then $Z^2 = Z$ and

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**Experiment:** Get coupons at random from $n$ until collect all $n$ coupons.
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**Outcomes:** \{123145..., 56765...\}
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**Experiment:** Get coupons at random from $n$ until collect all $n$ coupons.

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**Random Variable:** $X$ - length of outcome.

Before: $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$.

Today: $E[X]$?
Experiment: Get coupons at random from $n$ until collect all $n$ coupons.

Outcomes: $\{123145..., 56765...\}$

Random Variable: $X$ - length of outcome.

Before: $Pr[X \geq n \ln 2n] \leq \frac{1}{2}$.

Today: $E[X]$?
Time to collect coupons

\( X \)-time to get \( n \) coupons.
Time to collect coupons

$X$ - time to get $n$ coupons.

$X_1$ - time to get first coupon.
Time to collect coupons

$X$ - time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. 
Time to collect coupons

$X$ - time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$. 
Time to collect coupons

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$X_2$ - time to get second coupon after getting first.
Time to collect coupons

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$Pr[\text{“get second coupon”|“got milk”}]$
Time to collect coupons

$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”|“got milk first coupon”}] = \frac{n-1}{n}$
Time to collect coupons

$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”|\text{“got milk first coupon”}] = \frac{n-1}{n}$

$E[X_2]$?
Time to collect coupons

$X$-time to get $n$ coupons.
$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.
$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”} | \text{“got first coupon”}] = \frac{n-1}{n}$

$E[X_2]$? Geometric
Time to collect coupons

- time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$$Pr[\text{“get second coupon”}\mid \text{“got milk first coupon”}] = \frac{n-1}{n}$$

$E[X_2]$? Geometric!
Time to collect coupons

$X$-time to get $n$ coupons.

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$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{\rho} =$
Time to collect coupons

$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”|“got milk first coupon”}] = \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{\rho} = \frac{1}{\frac{n-1}{n}}$
Time to collect coupons

$X$-time to get $n$ coupons.

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$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{n-1} = \frac{n}{n-1}$. 
Time to collect coupons

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$E[X_2]$? Geometric !!! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n}{n-1}} = \frac{n}{n-1}$.

$Pr[\text{“getting $i^{th}$ coupon”}\mid \text{“got $i-1^{st}$ coupons”}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$
Time to collect coupons

$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”|“got milk first coupon”}] = \frac{n-1}{n}$

$E[X_2]$? Geometric !! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}.$

$Pr[\text{“getting $i$th coupon|“got $i-1$rst coupons”}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i]$
Time to collect coupons

$X$-time to get $n$ coupons.

$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

$X_2$ - time to get second coupon after getting first.

$Pr[\text{“get second coupon”|“got milk first coupon”}] = \frac{n-1}{n}$

$E[X_2]$? Geometric !!! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

$Pr[\text{“getting $i^{th}$ coupon|“got $i-1^{rst}$ coupons”}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p}$
Time to collect coupons

$X$-time to get $n$ coupons.
$X_1$ - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.
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$Pr[“getting ith coupon”|“got i – 1rst coupons”] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

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Time to collect coupons

\(X\)-time to get \(n\) coupons.

\(X_1\) - time to get first coupon. Note: \(X_1 = 1\). \(E(X_1) = 1\).

\(X_2\) - time to get second coupon after getting first.

\(Pr[\text{“get second coupon”} | \text{“got milk first coupon”}] = \frac{n-1}{n}\)

\(E[X_2]? \text{ Geometric ! ! !} \implies E[X_2] = \frac{1}{p} = \frac{1}{n-1} = \frac{n}{n-1} \cdot \)

\(Pr[\text{“getting \(i\)th coupon”} | \text{“got \(i-1\)rst coupons”}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}

\(E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.\)
**Time to collect coupons**

$X$-time to get $n$ coupons.

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$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$E[X] = E[X_1] + \cdots + E[X_n] =$
Time to collect coupons

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$\Pr[\text{"getting $i$th coupon"|\"got $i-1$rst coupons\}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \ldots, n.$

$E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$
Time to collect coupons

$X$-time to get $n$ coupons.

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$E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$

$= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n)$
Time to collect coupons

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$Pr[“\text{getting } i\text{th coupon”} | “\text{got } i-1\text{rst coupons”}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

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$$E[X] = E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$$

$$= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma)$$
Review: Harmonic sum

\[ H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} \, dx = \ln(n). \]
Review: Harmonic sum

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A good approximation is

\[ H(n) \approx \ln(n) + \gamma \] where \( \gamma \approx 0.58 \) (Euler-Mascheroni constant).
Harmonic sum: Paradox

Consider this stack of cards (no glue!):
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If each card has length 2, the stack can extend $H(n)$ to the right of the table.
Harmonic sum: Paradox

Consider this stack of cards (no glue!):

If each card has length 2, the stack can extend $H(n)$ to the right of the table. As $n$ increases, you can go as far as you want!
par·a·dox

/ˈpərə,ˌdæks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.

"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"

  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More

- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"
Stacking

The cards have width 2. Induction shows that the center of gravity after \( n \) cards is \( H(n) \) away from the right-most edge.

\[
H(n) = 1 + \frac{1}{2}
\]

\[
H(n + 1) = \frac{nx}{1 - x} = \frac{1}{n + 1}
\]

\[
x = \frac{1}{n + 1}
\]
Stacking

The cards have width 2.

Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.

$n x = 1 - x$

$\Rightarrow x = 1/(n + 1)$
The cards have width 2. Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.
Simeon Poisson

The Poisson distribution is named after:
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Equal Time: B. Geometric

The geometric distribution is named after:
The geometric distribution is named after:

B. Geometric (b. 300 BC)
Equal Time: B. Geometric

The geometric distribution is named after:

I could not find a picture of D. Binomial, sorry.
Experiment: flip a coin $n$ times. The coin is such that $Pr[H] = \lambda/n$.
Random Variable: $X$ - number of heads.
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Poisson
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**Poisson Distribution** is distribution of $X$ “for large $n$.”
Poisson

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We used $(1 - a/n)^n \approx e^{-a}$.
Poisson Distribution: Definition and Mean

**Definition** Poisson Distribution with parameter \( \lambda > 0 \)

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X = P(\lambda) \iff Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.
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Definition Poisson Distribution with parameter $\lambda > 0$

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Fact: $E[X] = \lambda$.

Proof:

$$E[X] = \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda}.$$
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Poisson Distribution: Variance

Recall: $X = P(\lambda) \Leftrightarrow Pr[X = n] = \frac{\lambda^n}{n!} e^{-\lambda}, n \geq 0.$
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Hence,

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Thus,

$$E[X^2] = \lambda^2 + E[X] = \lambda^2 + \lambda.$$
Poisson Distribution: Variance

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Consequently,

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Review: Distributions
**Review: Distributions**

**Geometric Distribution**
- Probability mass function: $(1 - p)^{x-1}p$
- Mean: $\frac{1}{p}$
- Variance: $\frac{1-p}{p^2}$

**Poisson Distribution**
- Probability mass function: $\frac{(\lambda^k/k!)}{e^{-\lambda}}$
- Mean: $\lambda$
- Variance: $\lambda$

**Binomial Distribution PDF**
- Probability mass function: $\binom{n}{k}p^k(1-p)^{n-k}$
- Mean: $np$
- Variance: $np(1-p)$
Summary

Variance, Geometric, time to collect coupons; Poisson
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Variance, Geometric, time to collect coupons; Poisson

- $\text{var}[a + bX] = b^2 \text{var}[X]$;
Summary

Variance, Geometric, time to collect coupons; Poisson

- \( \text{var}[a + bX] = b^2 \text{var}[X] \);  
- \( X, Y \) independent \( \Rightarrow \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \);
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Summary

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- Time to collect all of \( n \) coupons: \( nH(n) \approx n(\ln(n) + 0.58) \);
Summary

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- Intuition: Last coupon takes $1/n$ to collect!
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- Remember: $E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$.
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  - \( G(p) : E[X] = 1/p, \text{var}[X] = (1-p)/p^2 \);
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Summary

Variance, Geometric, time to collect coupons; Poisson

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  - \( P(\lambda) : E[X] = \lambda, \text{var}[X] = \lambda \)