1. The natural numbers.
2. 5 year old Gauss.
3. ..and Induction.
4. Simple Proof.
5. Two coloring map
7. Horses with one color...
8. Try this at home.
The naturals.
The naturals.
The naturals.

0, 1, 2, 3, ...
The naturals.

0, 1,
The naturals.

0, 1, 2,
The naturals.
0, 1, 2, 3,
The naturals. 

0, 1, 2, 3, ...

\[ 0, 1, 2, 3, ... \]
The naturals.

0, 1, 2, 3, ... , n,
The naturals.

0, 1, 2, 3, \ldots, n, n+1,
The naturals.

0, 1, 2, 3, ...
..., n, n+1, n+2, n+3,
The naturals.

0, 1, 2, 3,
..., n, n+1, n+2, n+3, ...

0
1
2
3
n
n+1
n+2
n+3
A formula.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's \( \frac{100 \times 101}{2} \) or 5050!
Teacher: Hello class.
A formula.

Teacher: Hello class.
Teacher:
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's \( \frac{(100)(101)}{2} \)
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's \( \frac{(100)(101)}{2} \) or 5050!
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\)
Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Proof by Induction.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Proof by Induction.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?
Gauss and Induction

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Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i\]
Gauss and Induction

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Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1)\]
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \) Proof?

Idea: assume predicate for \( n = k \). \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

Is predicate true for \( n = k + 1 \)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1
\]
Gauss and Induction

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Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]
Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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\]

How about \(k + 2\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!
Gauss and Induction

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How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof?
Gauss and Induction

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How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.
Gauss and Induction

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! **Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere.
Gauss and Induction

Child Gauss: $\forall n \in \mathbb{N} (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate for $n = k$. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate true for $n = k + 1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k + 2$. Same argument starting at $k + 1$ works! Induction Step.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}$
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[ \sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}. \]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)
Gauss and Induction

Child Gauss: $(\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate for $n = k$. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate true for $n = k + 1$?

$$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.$$  

How about $k + 2$. Same argument starting at $k + 1$ works! **Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}$ **Base Case.**

Statement is true for $n = 0$

   plus inductive step
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

    plus inductive step \(\implies\) true for \(n = 1\)
Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! 

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

- plus inductive step \(\implies\) true for \(n = 1\)
- plus inductive step
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

- plus inductive step \(\implies\) true for \(n = 1\)
- plus inductive step \(\implies\) true for \(n = 2\)
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \) Proof?

Idea: assume predicate for \( n = k \). \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

Is predicate true for \( n = k + 1 \)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \( k + 2 \). Same argument starting at \( k + 1 \) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \( \sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2} \) **Base Case.**

Statement is true for \( n = 0 \)

\[
\text{plus inductive step} \implies \text{true for } n = 1
\]

\[
\text{plus inductive step} \implies \text{true for } n = 2
\]

\[
\ldots
\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! **Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2} \) **Base Case.**

Statement is true for \(n = 0\)

- plus inductive step \(\implies\) true for \(n = 1\)
- plus inductive step \(\implies\) true for \(n = 2\)

\[\ldots\]

- true for \(n = k\)

\[\implies\] true for \(n = k\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! **Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)
  plus inductive step \(\implies\) true for \(n = 1\)
  plus inductive step \(\implies\) true for \(n = 2\)
  
  \[\vdots\]
  true for \(n = k\) \(\implies\) true for \(n = k + 1\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

plus inductive step \(\implies\) true for \(n = 1\)

plus inductive step \(\implies\) true for \(n = 2\)

\[\ldots\]

true for \(n = k\) \(\implies\) true for \(n = k + 1\)

\[\ldots\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

plus inductive step \(\implies\) true for \(n = 1\)

plus inductive step \(\implies\) true for \(n = 2\)

\[
\cdots \quad \text{true for } n = k \implies \text{true for } n = k + 1 \quad \cdots
\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^n i = \frac{n(n+1)}{2})\)  Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^k i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?
\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^k i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^1 i = 1 = \frac{(1)(2)}{2} \)  Base Case.

Statement is true for \(n = 0\)
   plus inductive step \(\implies\) true for \(n = 1\)
   plus inductive step \(\implies\) true for \(n = 2\)

\[\vdots\]
   true for \(n = k\) \(\implies\) true for \(n = k + 1\)

\[\vdots\]

Predicate True for all natural numbers!

**Proof by Induction.**
Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$
Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \( n \), \( 1 + 2 \cdots n = \frac{n(n+1)}{2} \).
Induction

The canonical way of proving statements of the form

\[(\forall k \in N)(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \(n \in N\), \(n^3 - n\) is divisible by 3.
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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The basic form
Induction

The canonical way of proving statements of the form

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- For all natural numbers \( n \), \( 1 + 2 \cdots n = \frac{n(n+1)}{2} \).
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The basic form

- Prove \( P(0) \). “Base Case”. 
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k + 1)$
Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
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The basic form

- Prove \(P(0)\). “Base Case”.
- \(P(k) \implies P(k+1)\)
  - Assume \(P(k)\), “Induction Hypothesis”
The canonical way of proving statements of the form 

\[(\forall k \in N)(P(k))\]

- For all natural numbers $n$, $1 + 2 \cdot \cdots \cdot n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k + 1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k + 1)$. “Induction Step.”
Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
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- Prove \(P(0)\). “Base Case”.
- \(P(k) \implies P(k+1)\)
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  - Prove \(P(k+1)\). “Induction Step.”

\(P(n)\) true for all natural numbers \(n\)!!!
Induction

The canonical way of proving statements of the form

\((\forall k \in \mathbb{N})(P(k))\)

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \(n \in \mathbb{N}\), \(n^3 - n\) is divisible by 3.
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The basic form

- Prove \(P(0)\). “Base Case”.
- \(P(k) \implies P(k + 1)\)
  - Assume \(P(k)\), “Induction Hypothesis”
  - Prove \(P(k + 1)\). “Induction Step.”

\(P(n)\) true for all natural numbers \(n\)!!
Get to use \(P(k)\) to prove \(P(k + 1)\)!
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers $n$!!!
Get to use $P(k)$ to prove $P(k+1)$!!
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k + 1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k + 1)$. “Induction Step.”

$P(n)$ true for all natural numbers $n$!!
Get to use $P(k)$ to prove $P(k + 1)$!!!
The canonical way of proving statements of the form

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An visualization: an infinite sequence of dominos.

Prove they all fall down;
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- $P(0) = \text{“First domino falls”}$
An visualization: an infinite sequence of dominos.

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An visualization: an infinite sequence of dominos.

Prove they all fall down;

- $P(0) = \text{“First domino falls”}$
- $(\forall k) \ P(k) \iff P(k+1)$: 
  “$k$th domino falls implies that $k+1$st domino falls”
Climb an infinite ladder?
Climb an infinite ladder?

Your favorite example of forever. or the integers...
Climb an infinite ladder?

\[ P(0) \]

\[ P(1) \]

\[ P(2) \]

\[ P(3) \]

\[ P(n) \]

\[ P(n+1) \]

\[ P(\cdot) \]

\[ P(\cdot) \]

\[ P(k) = \Rightarrow P(k+1) \]

\[ (\forall n \in \mathbb{N}) P(n) \]

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\[ P(0) \]

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\( \forall n \in \mathbb{N} \)
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\[ P(0) \]

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\[ \vdots \]

\[ P(n) \]

\[ P(n+1) \]

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\[ \forall n \in \mathbb{N} \]

\[ P(n) \]

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\[
P(0) \quad P(k) \implies P(k + 1)
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Climb an infinite ladder?

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\[ P(n) \]

\[ P(0) \Rightarrow P(k+1) \]

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Your favorite example of forever...
or the integers...

\[ P(0) \]

\[ P(k) \implies P(k + 1) \]
Climb an infinite ladder?

\[ P(0) \implies P(n+1) \]

\[
\begin{align*}
P(k) & \implies P(k + 1) \\
(\forall n \in N) P(n)
\end{align*}
\]
Climb an infinite ladder?

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Climb an infinite ladder?

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Your favorite example of forever..or the integers...
Simple induction proof.

**Theorem:** For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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Induction Hypothesis: \( 1 + \cdots + n = \frac{n(n+1)}{2} \)

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1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n^2 + n + 2(n+1)}{2} = \frac{n^2 + 3n + 2}{2}
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\[
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\( P(n+1)! \)
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$(\forall n \in \mathbb{N}) (P(n) \implies P(n+1))$. \qed
Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.
Two color theorem: example.

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Fact: Swapping red and blue gives another valid colors.
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Fact: Swapping red and blue gives another valid colors.
Two color theorem: proof illustration.

Base Case.

1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.
   (Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \Rightarrow P(k+1)$. 
Two color theorem: proof illustration.

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Tiling Cory Hall Courtyard.
Tiling Cory Hall Courtyard.

A

B

C

D

E
Tiling Cory Hall Courtyard.
Tiling Cory Hall Courtyard.
Tiling Cory Hall Courtyard.
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Summary: principle of induction.

\( P(0) \)
Summary: principle of induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1))))\]
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\[(P(0) \land ((\forall k \in \mathbb{N})(P(k) \implies P(k + 1)))) \implies (\forall n \in \mathbb{N})(P(n))\]
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Variations:
\[(P(0) \land ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))\]
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\[(P(1) \land ((\forall n \in \mathbb{N})(n \geq 1 \land P(n)) \implies P(n+1))))\]

Statement to prove:

\[P(n)\] for \[n\] starting from \[n_0\].

Base Case: Prove \[P(n_0)\].

Ind. Step: Prove.

For all values, \[n \geq n_0\], \[P(n) = \implies P(n+1)\].

Statement is proven!
Summary: principle of induction.

\((P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\)

Variations:
\((P(0) \land ((\forall n \in N)(P(n) \implies P(n + 1)))) \implies (\forall n \in N)(P(n))\)

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