CS70: Jean Walrand: Lecture 36.

Gaussian and CLT

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1. Review of continuous probability
2. Motivation for Gaussian
3. Gaussian
4. CLT

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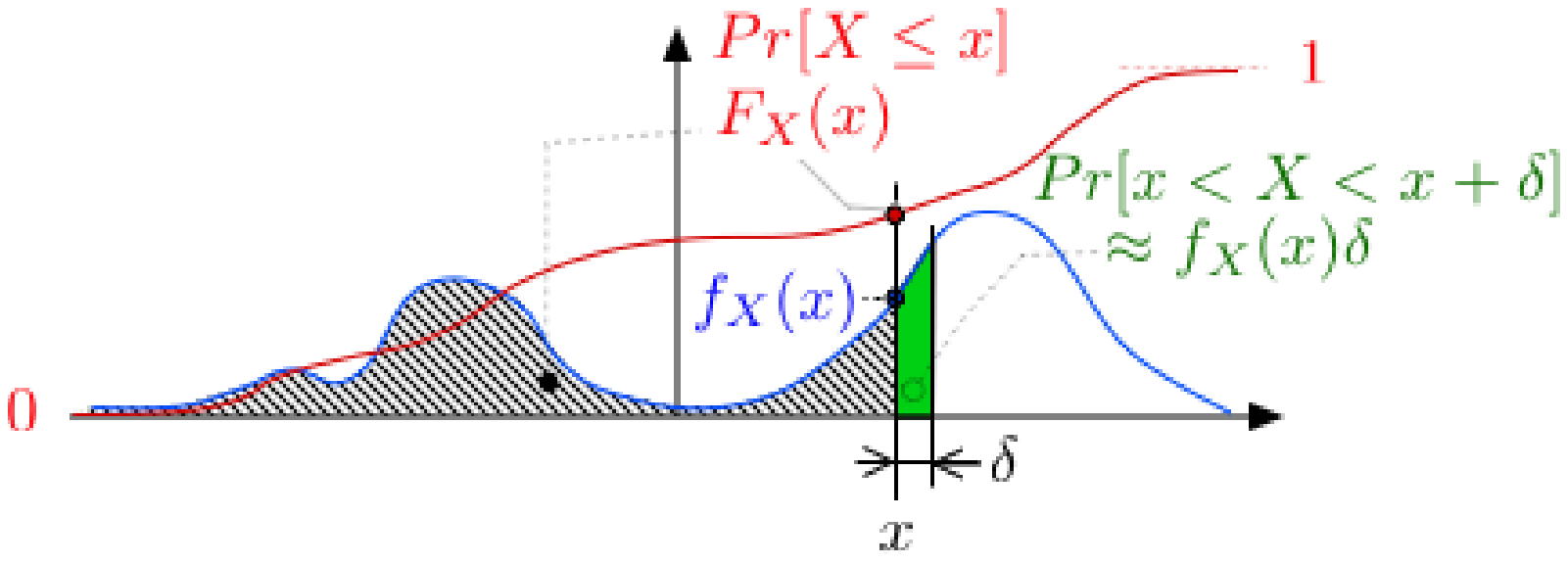
Recall that $Pr[X \in (x, x+\delta)] \approx f_X(x)\delta$. Think of X taking discrete values $n\delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with $Pr[X = n\delta] = f_X(n\delta)\delta$.

A Picture



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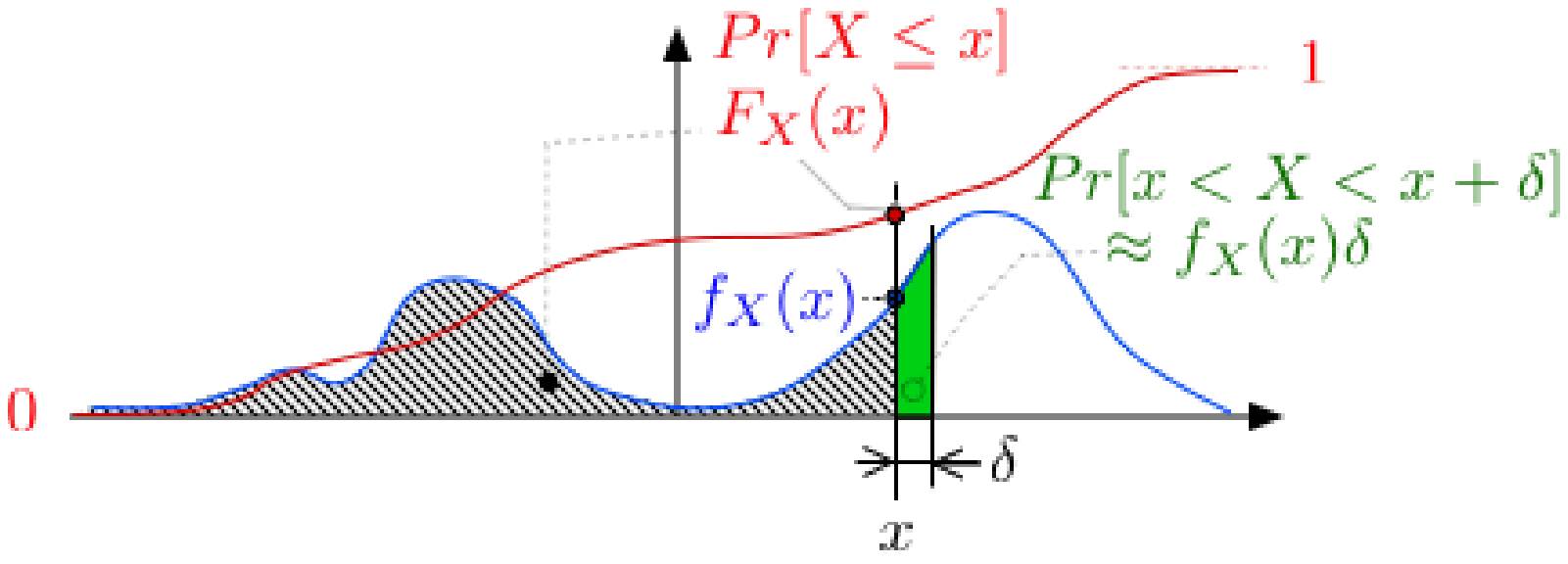

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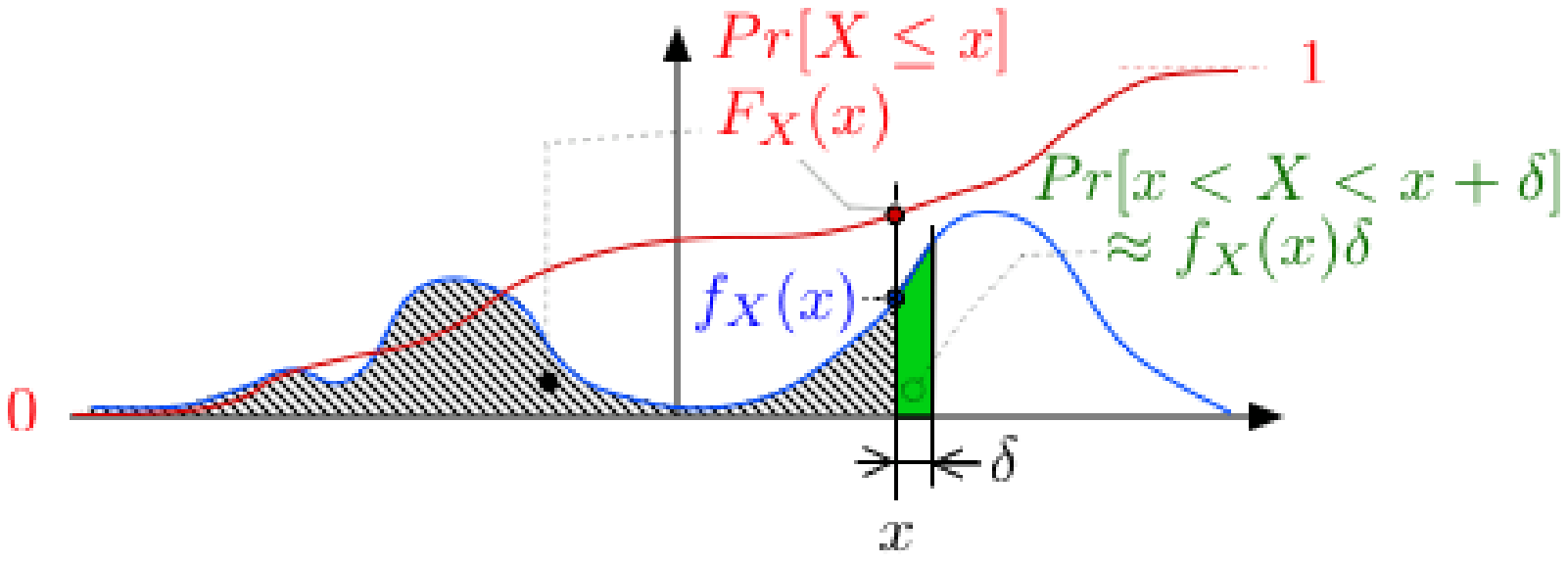

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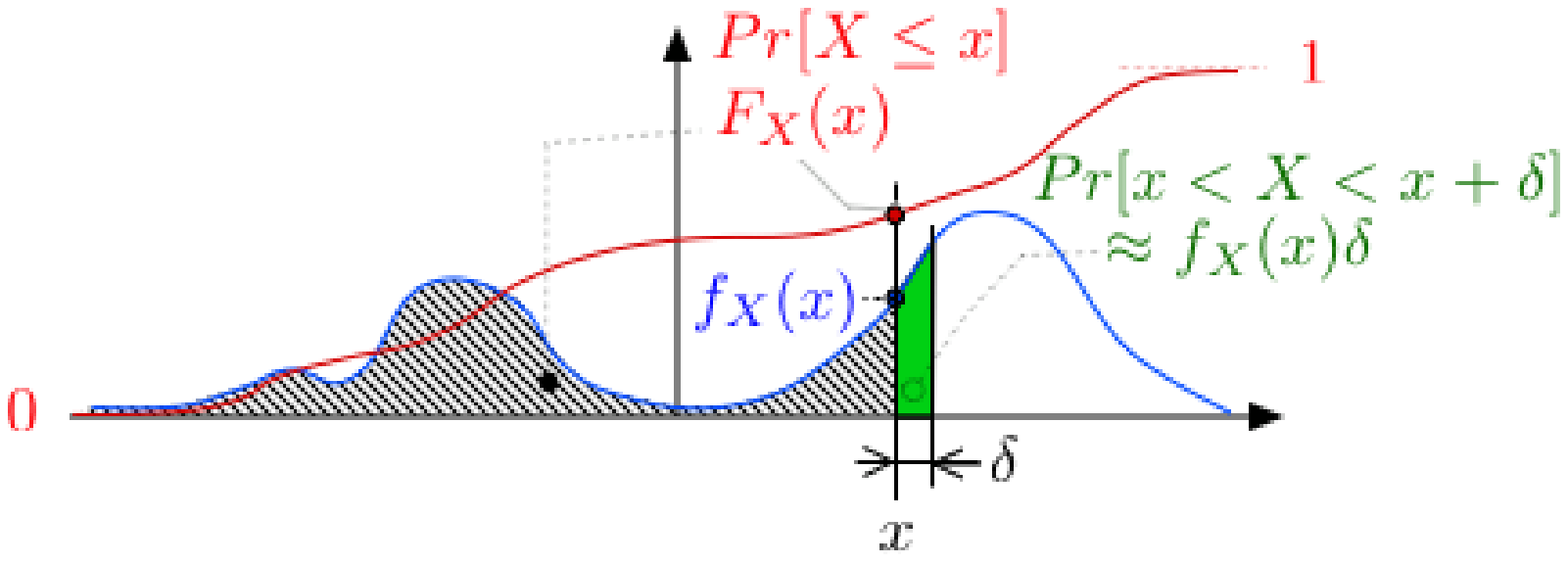

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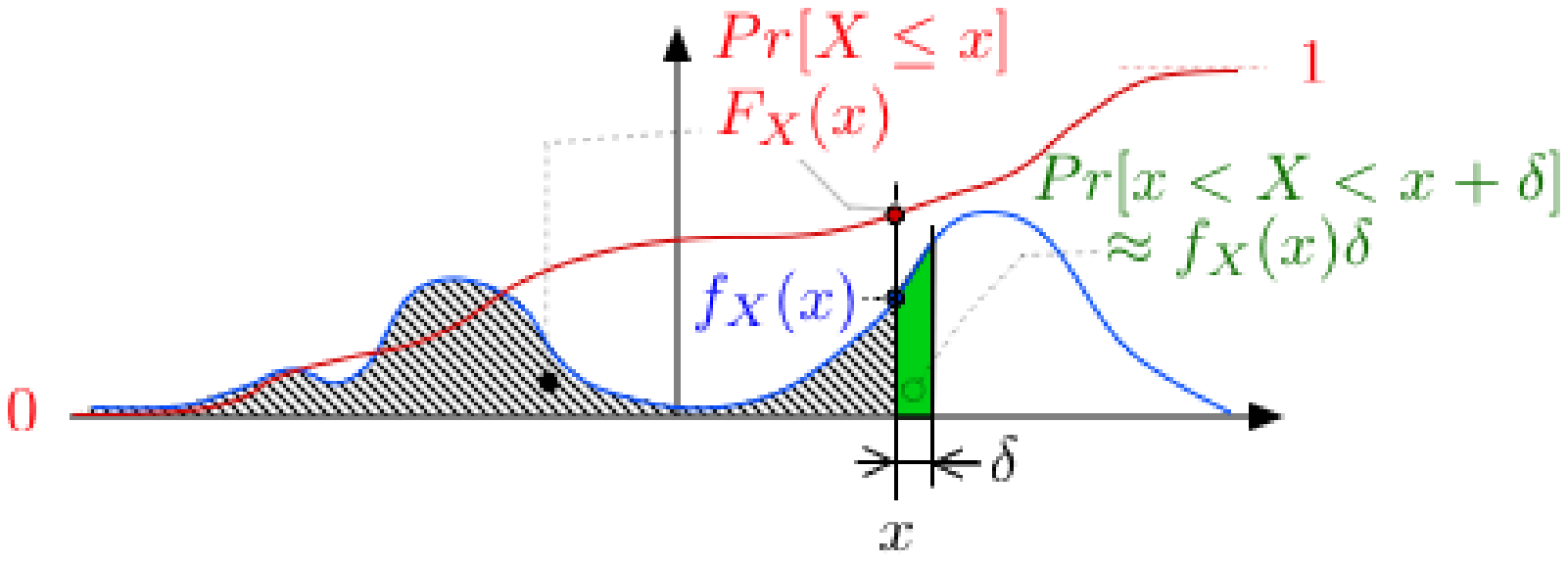

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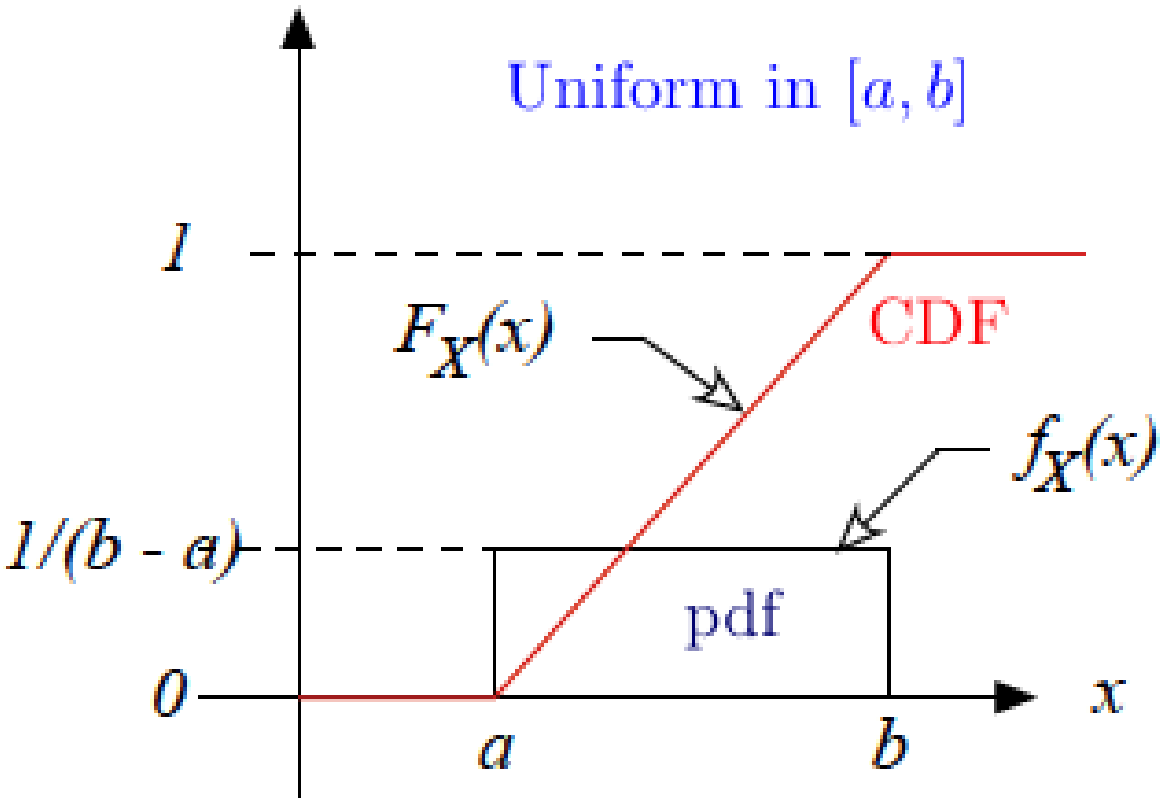
Example: $U[a, b]$



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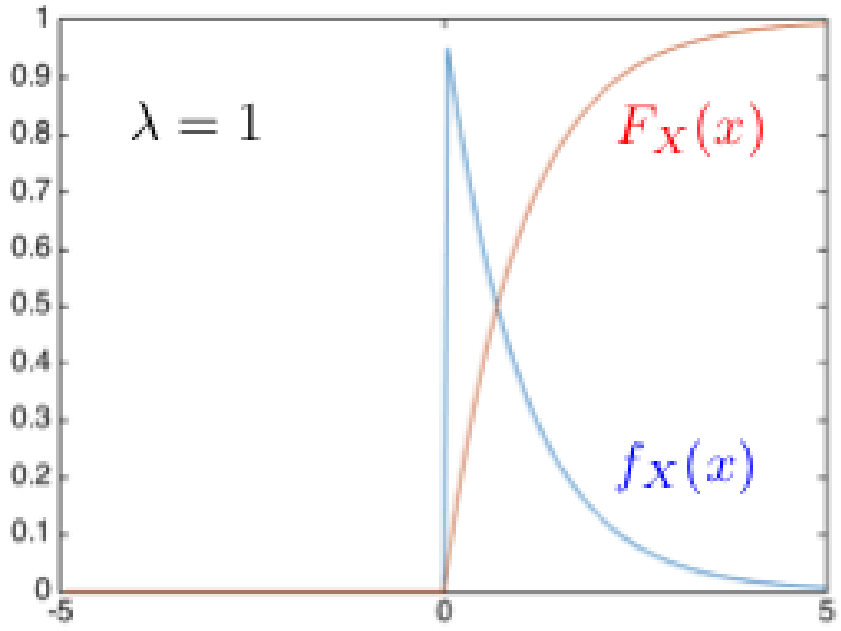
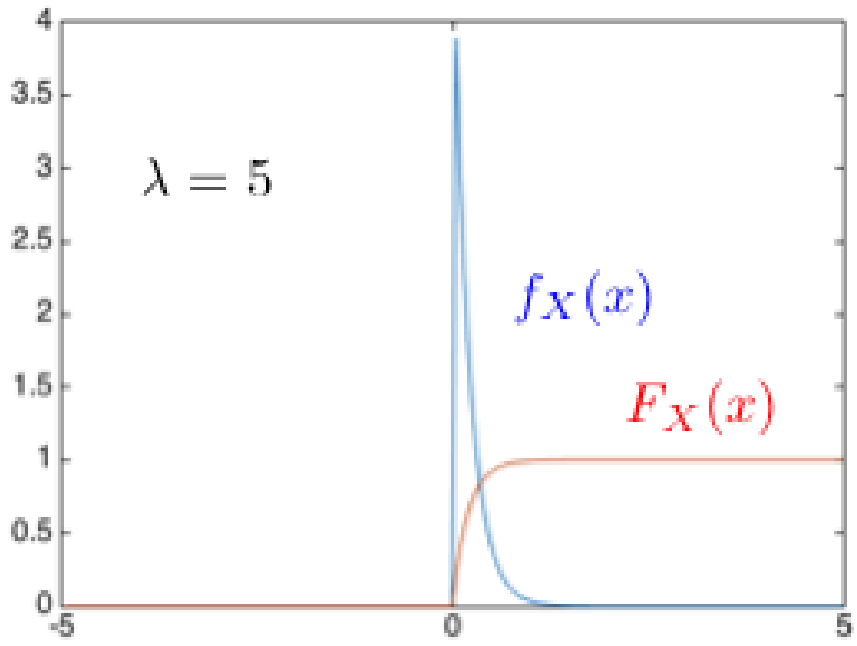
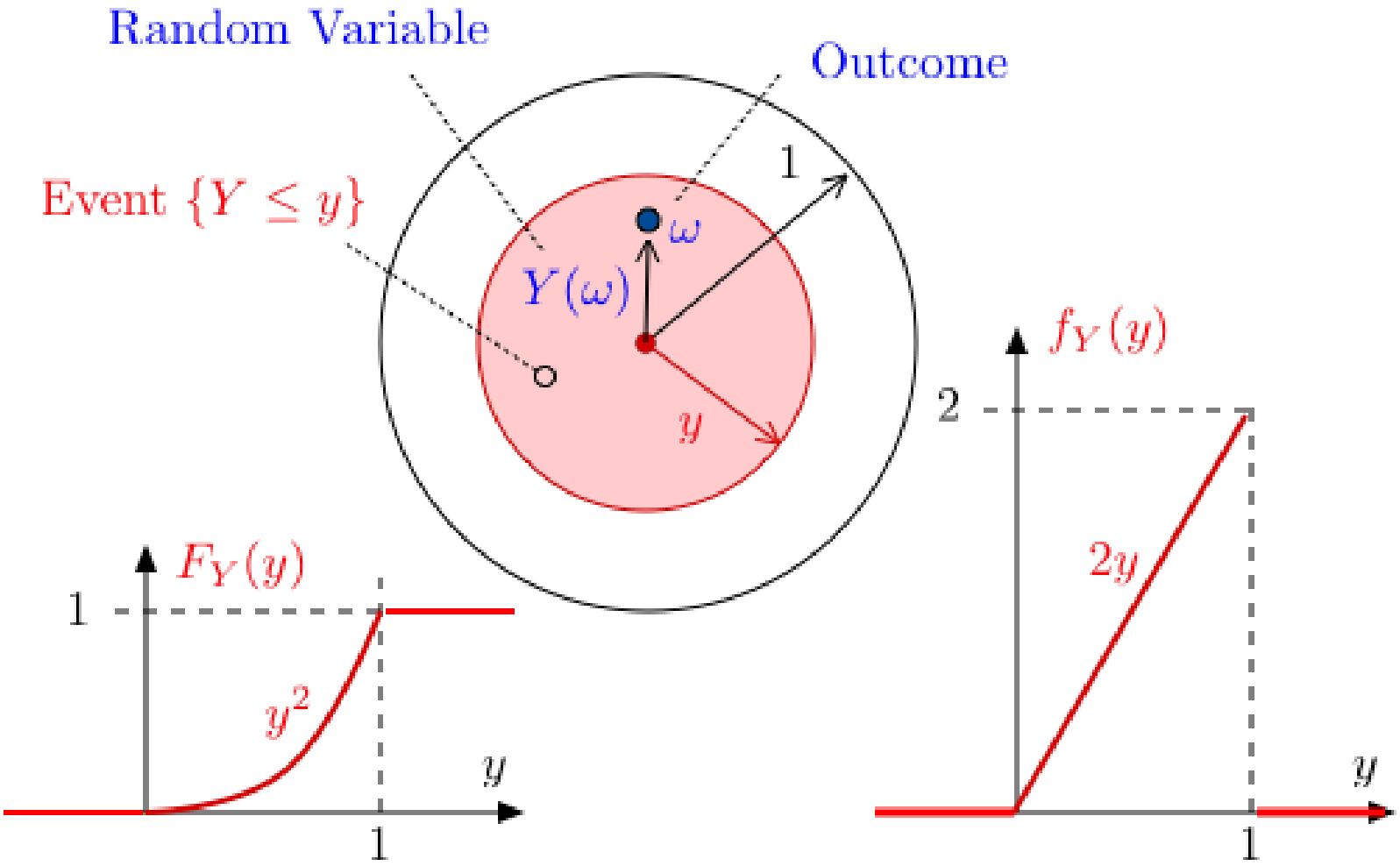




Shooting in a circle



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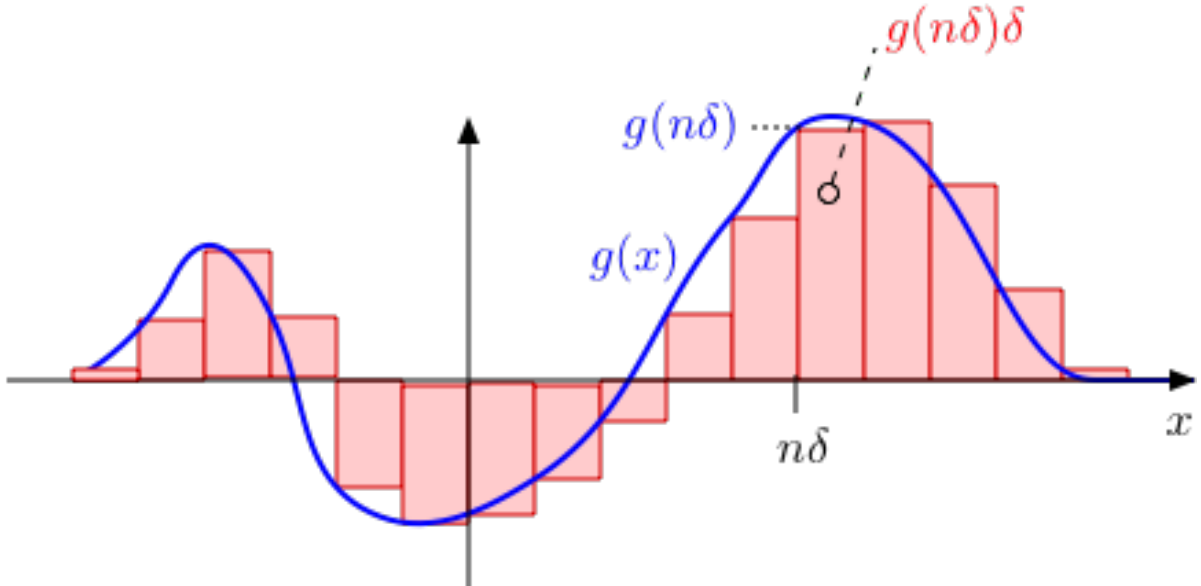

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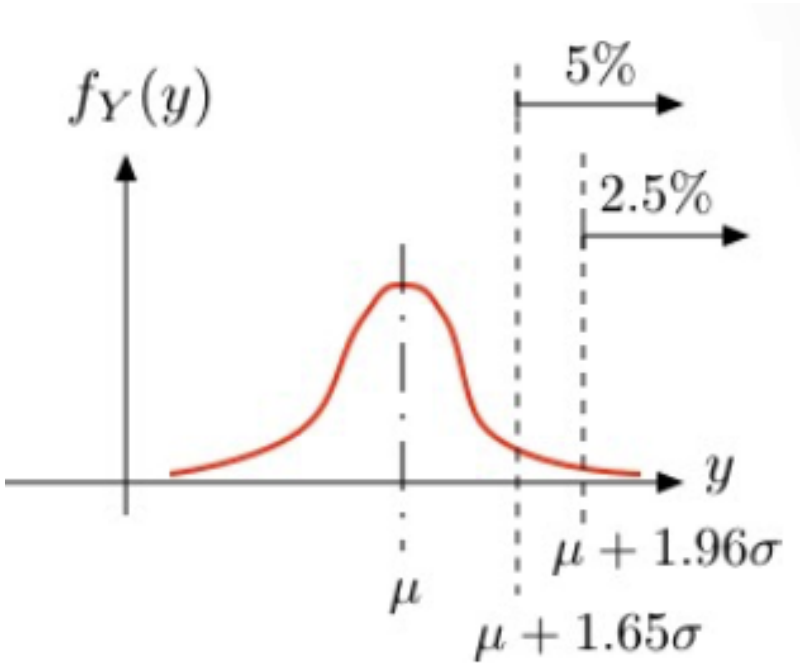

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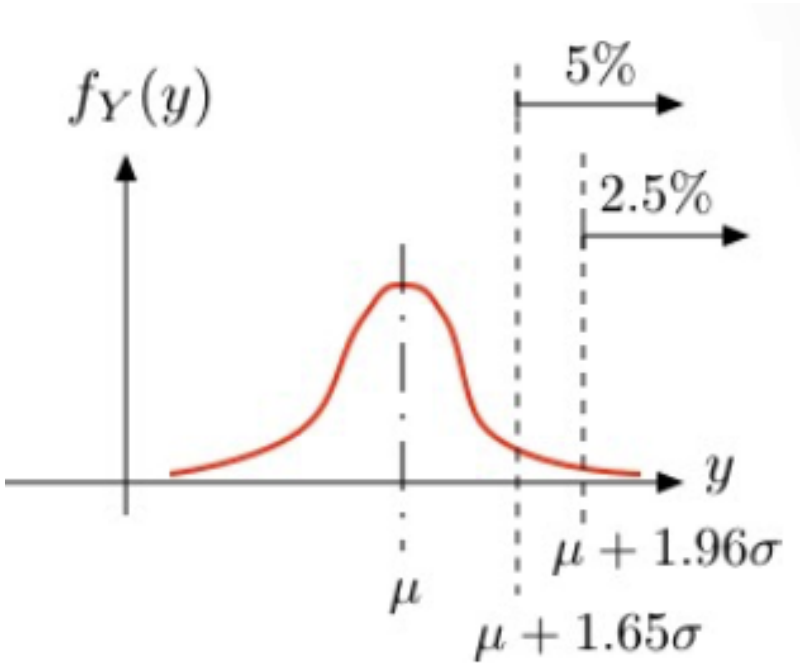

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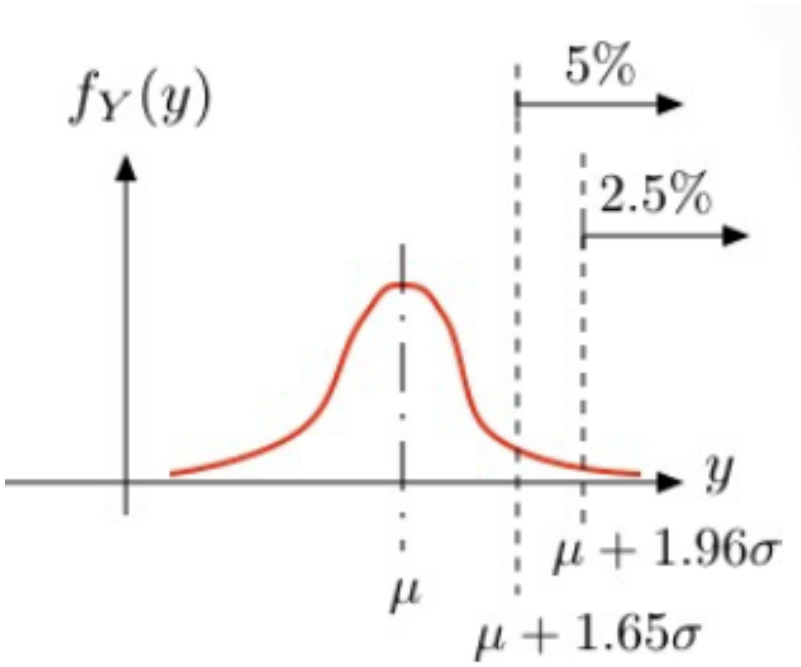

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Central limit theorem: As n goes to infinity the distribution of A'_n approaches the standard normal distribution.

$$Pr[A'_n \leq \alpha] \to \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\alpha} e^{-x^2/2} dx.$$

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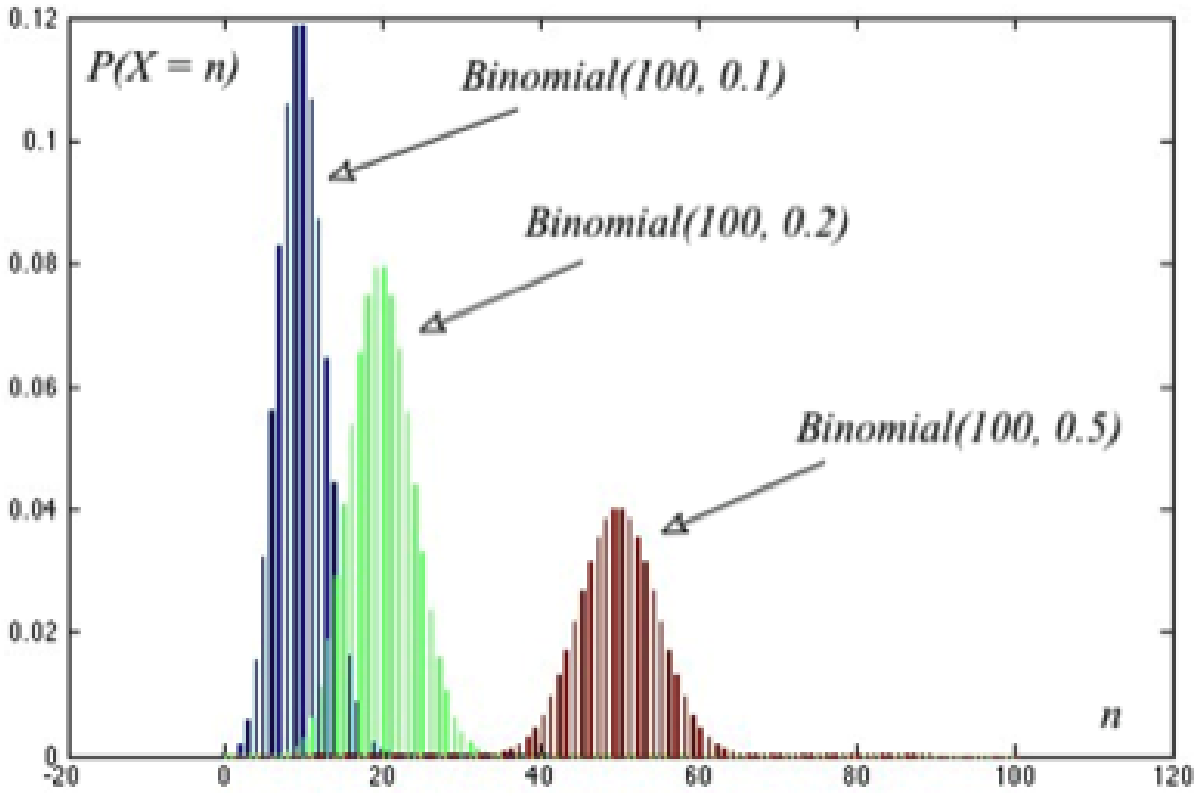

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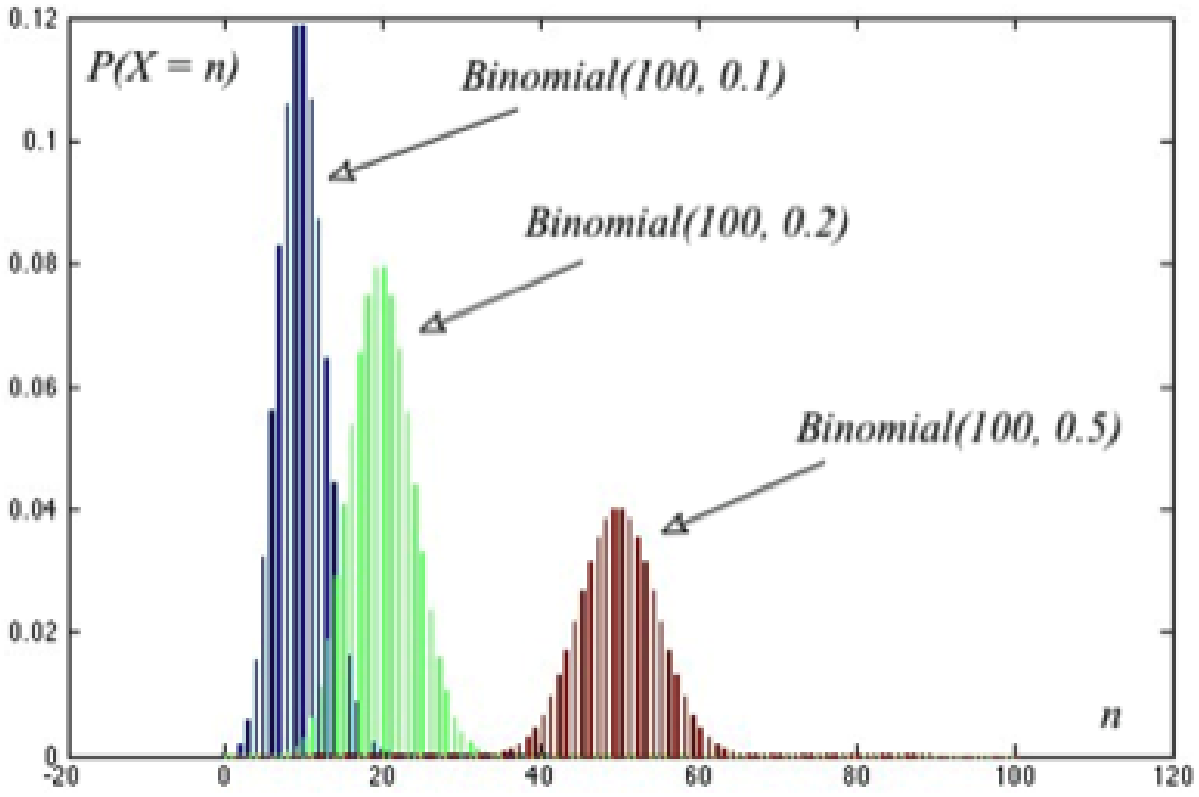

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Summary

$\boxed{\text{Gaussian and CLT}}$

1. Gaussian: $\mathscr{N}(\mu, \sigma^2) : f_X(x) = \ldots$ "bell curve"
2. CLT: X_n i.i.d. $\implies \frac{A_n - \mu}{\sigma/\sqrt{n}} \to \mathscr{N}(0,1)$
3. CI: $[A_n - 2\frac{\sigma}{\sqrt{n}}, A_n + 2\frac{\sigma}{\sqrt{n}}] = 95\%$-CI for μ.