CS70: Jean Walrand: Lecture 39.

Final - Probability Review

- True or False
- Some Key Results
- Sample Problems
- Common Mistakes

True or False

- Ω and A are independent. True
- $Pr[A \cap B] = Pr[A] + Pr[B] - Pr[A \cup B]$. True
- $Pr[A \setminus B] \geq Pr[A] - Pr[B]$. True
- $X_1, \ldots, X_n$ i.i.d. $\implies var(\frac{X_1 + \cdots + X_n}{n}) = var(X_1)$. False: $\times \frac{1}{n}$
- $Pr[|X - a| \geq b] \leq \frac{E[(X-a)^2]}{b^2}$. True
- $X_1, \ldots, X_n$ i.i.d. $\implies \frac{X_1 + \cdots + X_n - nE[X_1]}{n\sigma(X_1)} \to \mathcal{N}(0,1)$. False: $\sqrt{n}$
- $X = Expo(\lambda) \implies Pr[X > 5 | X > 3] = Pr[X > 2]$. True: $\frac{\exp\{-\lambda 5\}}{\exp\{-\lambda 3\}} = \exp\{-\lambda 2\}$.

Correct or not?

- $[A_n - 2\sigma\frac{1}{n}, A_n + 2\sigma\frac{1}{n}] = 95\%$-CI for μ. No
- $[A_n - 2\sigma\frac{1}{\sqrt{n}}, A_n + 2\sigma\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. Yes
- If $0.3 < \sigma < 3$, then $[A_n - 0.6\frac{1}{\sqrt{n}}, A_n + 0.6\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. No
- If $0.3 < \sigma < 3$, then $[A_n - 6\frac{1}{\sqrt{n}}, A_n + 6\frac{1}{\sqrt{n}}] = 95\%$-CI for μ. Yes

Match Items

[1] $Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$.

[2] $Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}$

[3] $Pr[X \geq a] \leq \min_{\theta > 0} \frac{E[e^{\theta X}]}{e^{\theta a}}$

[4] $g(\cdot)$ convex $\Rightarrow E[g(X)] \geq g(E[X])$

[5] $E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$.

[6] $\sum_y y Pr[Y = y | X = x]$

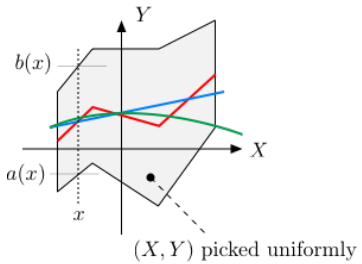
[7] $Pr[|\frac{X_1 + \cdots + X_n}{n} - E[X_1]| \geq \varepsilon] \to 0$,

[8] $E[(Y - E[Y|X])h(X)] = 0$.

- Chernoff (3)
- WLLN (7)
- Jensen (4)
- MMSE (6)
- Projection property (8)
- Chebyshev (2)
- LLSE (5)
- Markov (1)

Conditional Expectation

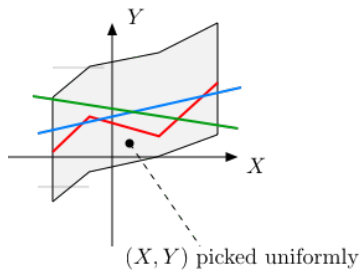
Which is $E[Y|X]$? Blue, red or green?



Answer: Red.
Given $X = x$, $Y = U[a(x), b(x)]$. Thus, $E[Y|X = x] = \frac{a(x)+b(x)}{2}$.

Linear Regression

Which is $L[Y|X]$? Blue, red or green?



Answer: Blue.
Cannot be red (not a straight line).
Cannot be green: X and Y are clearly positively correlated.

LLSE

A bag has n red and n blue balls. You pick two balls (no replacement). Let $X = 1$ if ball 1 is red and $X = -1$ otherwise. Define Y likewise for ball 2.
$\rightarrow$ Are X and Y positively, negatively, or un- correlated?
Clearly, negatively.
$\rightarrow$ Calculate $cov(X,Y)$.

$cov(X,Y) = E[XY] - E[X]E[Y]$

$E[X] = E[Y]$, by symmetry

$E[X] = 0$

$E[XY] = Pr[X = Y] - Pr[X \neq Y] = 2Pr[X = Y] - 1$

$Pr[X = Y] = (n-1)/(2n-1)$

E.g., if $X = +1$ = red, then Y is red w.p. $(n-1)/(2n-1)$

$E[XY] = 2(n-1)/(2n-1) - 1 = -1/(2n-1) = cov(X,Y)$.

$\rightarrow$ What is $L[Y|X]$? $L[Y|X] = -\frac{1}{2n-1}X$. Indeed, $var(X) = 1$, obviously!

CE

A bag has n red and n blue balls. You pick two balls (no replacement). Let $X = 1$ if ball 1 is red and $X = -1$ otherwise. Define Y likewise for ball 2.
$\rightarrow$ Calculate $E[Y|X]$.
Since X takes only two values, any $g(X)$ is linear in X.
Hence, $E[Y|X] = L[Y|X]$.

Alternatively, Let $\alpha = Pr[X = Y] = (n-1)(2n-1)$.Then,

$$E[Y|X = 1] = \alpha - (1-\alpha) = 2\alpha - 1,$$
$$E[Y|X = -1] = -\alpha + (1-\alpha) = 1 - 2\alpha.$$

Thus,

$$E[Y|X] = (2\alpha - 1)X = -\frac{1}{2n-1}X.$$

CE

Let X, Y, Z be i.i.d.
$\rightarrow$ Compute $E[X|X+Y+Z]$.
Answer: $(X+Y+Z)/3$.
Indeed,

$$E[X|X+Y+Z] = E[Y|X+Y+Z] = E[Z|X+Y+Z]$$

and these three quantities add up to

$$E[X+Y+Z|X+Y+Z] = X+Y+Z.$$

Continuous RV

Let $X = Expo(1)$ and $Y = Expo(2)$ be independent.
Let $Z = \max\{X,Y\}$. Calculate $E[Z]$.
Recall: $V = Expo(\lambda) \implies f_V(x) = \lambda \exp\{-\lambda x\} 1\{x \geq 0\}$.
Also, $Pr[V \leq x] = 1 - \exp\{-\lambda x\}$ for $x \geq 0$.
Moreover, $E[V] = \int_0^\infty x\lambda \exp\{-\lambda x\} dx = \lambda^{-1}, var(V) = \lambda^{-2}$.

For $z > 0$, one has

$$\begin{aligned} Pr[\max\{X,Y\} \leq z] &= Pr[X \leq z, Y \leq z] = Pr[X \leq z]Pr[Y \leq z]. \\ &= (1 - \exp\{-z\})(1 - \exp\{-2z\}) \\ &= 1 - \exp\{-z\} - \exp\{-2z\} + \exp\{-3z\}. \end{aligned}$$

Thus, for $z > 0$, taking the derivative,

$$f_Z(z) = \exp\{-z\} + 2\exp\{-2z\} - 3\exp\{-3z\}.$$

Hence,

$$E[Z] = \int_0^\infty z f_Z(z) dz = 1 + \frac{1}{2} - \frac{1}{3} = \frac{7}{6}.$$

Continuous RV

Let $X = Expo(1)$ and $Y = Expo(2)$ be independent.
Let $W = \min\{X,Y\}$. Calculate $E[W]$.
Recall: $V = Expo(\lambda) \implies f_V(x) = \lambda \exp\{-\lambda x\} 1\{x \geq 0\}$.
Also, $Pr[V \leq x] = 1 - \exp\{-\lambda x\}$ for $x \geq 0$.
Moreover, $E[V] = \int_0^\infty x\lambda \exp\{-\lambda x\} dx = \lambda^{-1}, var(V) = \lambda^{-2}$.

For $z > 0$, one has

$$\begin{aligned} Pr[\min\{X,Y\} \geq z] &= Pr[X \geq z, Y \geq z] = Pr[X \geq z]Pr[Y \geq z]. \\ &= \exp\{-z\}\exp\{-2z\} = \exp\{-3z\}. \end{aligned}$$

Thus, $W = Expo(3)$. Hence, $E[W] = \frac{1}{3}$.

Continuous RV and Bayes' Rule

W.p. $1/2$, X, Y are i.i.d. $Expo(1)$ and w.p. $1/2$, they are i.i.d. $Expo(3)$.

Calculate $E[Y|X = x]$.

Let B be the event that $X \in [x, x+\delta]$ where $0 < \delta \ll 1$.

Let A be the event that X, Y are $Expo(1)$.

Then,

$$\begin{aligned} Pr[A|B] &= \frac{(1/2)Pr[B|A]}{(1/2)Pr[B|A] + (1/2)Pr[B|\bar{A}]} = \frac{\exp\{-x\}\delta}{\exp\{-x\}\delta + 3\exp\{-3x\}\delta} \\ &= \frac{\exp\{-x\}}{\exp\{-x\} + 3\exp\{-3x\}} = \frac{e^{2x}}{3 + e^{2x}}. \end{aligned}$$

Now,

$$\begin{aligned} E[Y|X = x] &= E[Y|A]Pr[A|X = x] + E[Y|\bar{A}]Pr[\bar{A}|X = x] \\ &= 1 \times Pr[A|X = x] + (1/3)Pr[\bar{A}|X = x] ... = \frac{1 + e^{2x}}{3 + e^{2x}}. \end{aligned}$$

We used $Pr[Z \in [x, x+\delta]] \approx f_Z(x)\delta$ and given A one has $f_X(x) = \exp\{-x\}$ whereas given $\bar{A}$ one has $f_X(x) = 3\exp\{-3x\}$.

Rolling Dice

You roll a balanced die.

You start with \$1.00.

Every time you get a 6, your fortune is multiplied by 10.

Every time you do not get a 6, your fortune is divided by 2.

Let X_n be your fortune at the start of step n,

Calculate $E[X_n]$.

We have $X_1 = 1$. Also,

$$\begin{aligned} E[X_{n+1}|X_n] &= X_n \times [10\frac{1}{6} + 0.5 \times \frac{5}{6}] \\ &= \rho X_n, \rho = 10\frac{1}{6} + 0.5 \times \frac{5}{6} \approx 2.1. \end{aligned}$$

Hence,

$$E[X_{n+1}] = \rho E[X_n], n \geq 1.$$

Thus,

$$E[X_n] = \rho^{n-1}, n \geq 1.$$

Common Mistakes

- $\Omega = \{1,2,3\}$. Define X, Y with $cov(X,Y) = 0$ and X, Y not independent.

  Let $X = 0, Y = 1$. No: They are independent.

  Let

  $X(1) = -1, X(2) = 0, X(1) = 1, Y(1) = 0, Y(2) = 1, Y(3) = 0.$
- $3 \times 3.5 = 12.5$. No.
- $E[X^2] = E[X]^2$. No.
- $X = B(n,p) \implies var(X) = n^2 p(1-p)$. No.
- $E[X] = E[X|A] + E[X|\bar{A}]$. No.
- $\sum_{n=0}^{\infty} a^n = 1/a$. No.
- CS70 is difficult. No.
- I will do poorly on the final. No.
- Walrand is really weird. Probably!.

Thanks and Best Wishes!