True or False
▶ True or False
▶ Some Key Results
▶ Sample Problems
▶ Common Mistakes

Ω and A are independent. True


Pr[A] > Pr[B]. True

X₁, ..., Xₙ i.i.d. ⇒ var(X₁) = var(Xₙ). False: ×₁ⁿ

Pr[A \ B] ≥ Pr[A] − Pr[B]. True

exp{−λ5} exp{−λ3} = exp{−λ2}. True:

Correct or not?
▶ [A₁ - 2σ₁ √n , A₁ + 2σ₁ √n ] = 95%-CI for µ. No
▶ [A₁ - a₁, A₁ + a₁] = 95%-CI for µ. Yes
▶ If 0 < σ < 3, then [A₁ - 0.6 1√n , A₁ + 0.6 1√n ] = 95%-CI for µ. No
▶ If 0 < σ < 3, then [A₁ - 0.6 1√n , A₁ + 0.6 1√n ] = 95%-CI for µ. Yes

Match Items
▶ Chernoff (3)
▶ WLLN (7)
▶ Jensen (4)
▶ MMSE (6)
▶ Projection property (8)
▶ Chebyshev (2)
▶ LLSE (5)
▶ Markov (1)

Conditional Expectation

Which is E[Y|X]? Blue, red or green?

Answer: Red.

Given X = x, Y = U[a(x), b(x)]. Thus, E[Y|X = x] = \frac{a(x) + b(x)}{2}.

Linear Regression

Which is L[Y|X]? Blue, red or green?

Answer: Blue.

Cannot be red (not a straight line).

Cannot be green: X and Y are clearly positively correlated.

Answer: Blue.

Cannot be red (not a straight line).

Cannot be green: X and Y are clearly positively correlated.
LLSE

A bag has $n$ red and $n$ blue balls. You pick two balls (no replacement). Let $X = 1$ if ball 1 is red and $X = -1$ otherwise.

Define $Y$ likewise for ball 2.

→ Are $X$ and $Y$ positively, negatively, or uncorrelated?

Clearly, negatively.

→ Calculate $\text{cov}(X, Y)$.

\[
\text{cov}(X, Y) = E[XY] - E[X]E[Y]
\]

$E[X] = E[Y]$; by symmetry

$E[X] = 0$

$E[XY] = Pr[X = Y] - Pr[X \neq Y] = 2Pr[X = Y] - 1$

$Pr[X = Y] = (n-1)/(2n-1)$

E.g., if $X = +1$ is red, then $Y$ is red w/p.

Hence,

$E[XY] = 2(n-1)/(2n-1) - 1 = -1/(2n-1) = \text{cov}(X, Y)$.

→ What is $L[|X|]$? $L[|X|] = \frac{1}{n}\text{var}(X)$. Indeed, $\text{var}(X) = 1$, obviously!

Continuous RV

Let $X = \text{Exp}(1)$ and $Y = \text{Exp}(2)$ be independent.

Let $Z = \max(X, Y)$. Calculate $E[Z]$.

Recall: $V = \text{Exp}(\lambda) \implies f_V(x) = \lambda e^{-\lambda x}1\{x \geq 0\}$

Also, $Pr[V \leq x] = 1 - \exp(-\lambda x)$ for $x \geq 0$.

Moreover, $E[V] = \int_0^\infty \lambda x e^{-\lambda x} dx = \frac{1}{\lambda}$, $\text{var}(V) = \frac{1}{\lambda^2}$.

For $z > 0$, one has

\[
Pr[\max(X, Y) \leq z] = Pr[X \leq z, Y \leq z] = Pr[X \leq z]Pr[Y \leq z] = (1 - \exp[-\lambda z])(1 - \exp[-2\lambda z]) = 1 - \exp[-\lambda z] - \exp[-2\lambda z] + \exp[-3\lambda z].
\]

Thus, for $z > 0$, taking the derivative,

\[
f_Z(z) = \exp(-z) + 2\exp(-2z) - 3\exp(-3z).
\]

Hence,

\[
E[Z] = \int_0^\infty z f_Z(z) dz = 1/2 - 1/3 = \frac{1}{5}.
\]

CE

A bag has $n$ red and $n$ blue balls. You pick two balls (no replacement). Let $X = 1$ if ball 1 is red and $X = -1$ otherwise. Define $Y$ likewise for ball 2.

→ Calculate $E[Y|X]$.

Since $X$ takes only two values, any $g(X)$ is linear in $X$.

Hence, $E[Y|X] = L[Y|X]$.

Alternatively, Let $\alpha = Pr[X = Y] = (n-1)/(2n-1)$. Then,

\[
E[Y|X = 1] = \alpha - (1 - \alpha) = 2\alpha - 1,
\]

\[
E[Y|X = -1] = -\alpha + (1 - \alpha) = 1 - 2\alpha.
\]

Thus,

\[
E[Y|X] = (2\alpha - 1)X = -\frac{1}{2n-1}X.
\]

Continuous RV

Let $X = \text{Exp}(1)$ and $Y = \text{Exp}(2)$ be independent.

Let $W = \min(X, Y)$. Calculate $E[W]$.

Recall: $V = \text{Exp}(\lambda) \implies f_V(x) = \lambda e^{-\lambda x}1\{x \geq 0\}$

Also, $Pr[V \leq x] = 1 - \exp(-\lambda x)$ for $x \geq 0$.

Moreover, $E[V] = \int_0^\infty \lambda x e^{-\lambda x} dx = -1/\lambda$, $\text{var}(V) = 1/\lambda^2$.

For $z > 0$, one has

\[
Pr[\min(X, Y) \geq z] = Pr[X \geq z, Y \geq z] = Pr[X \geq z]Pr[Y \geq z] = \exp(-z) \exp(-2z) = \exp(-3z).
\]

Thus, $W = \text{Exp}(3)$. Hence, $E[W] = \frac{1}{3}$.

Continuous RV and Bayes’ Rule

W.p. 1/2, $X, Y$ are i.i.d. $\text{Exp}(1)$ and w.p. 1/2, they are i.i.d. $\text{Exp}(3)$.

Calculate $E[Y|X = x]$.

Let $B$ be the event that $X \in [x, x + \delta]$ where $0 < \delta \ll 1$.

Let $A$ be the event that $X, Y$ are $\text{Exp}(1)$.

Then,

\[
Pr[A|B] = \frac{1/2Pr[B|A]}{1/2Pr[B|A] + 1/2Pr[B|\bar{A}]} = \frac{\exp(-x)\delta}{\exp(-x)\delta + 3\exp(-3x)\delta} = \frac{\exp(-x)}{\exp(-x) + 3\exp(-3x)} = \frac{\exp(-x)}{3 + \exp(-2x)}.
\]

Now,

\[
E[Y|X = x] = E[Y|A]Pr[A|X = x] + E[Y|\bar{A}]Pr[\bar{A}|X = x] = \exp(-x) + 1\times \exp(-x) + 1/3\times \exp(-x) = \frac{1 + \exp(-x)}{3 + \exp(-2x)}.
\]

We used $Pr[Z \in [x, x + \delta]] = f_Z(x)\delta$ and given $A$ one has $f_Z(x) = \exp(-x)$ whereas given $\bar{A}$ one has $f_Z(x) = 3\exp(-3x)$.
Rolling Dice
You roll a balanced die.
You start with $1.00.
Every time you get a 6, your fortune is multiplied by 10.
Every time you do not get a 6, your fortune is divided by 2.
Let $X_n$ be your fortune at the start of step $n$.
Calculate $E[X_n]$.

We have $X_1 = 1$. Also,
\[
E[X_{n+1}|X_n] = X_n \times \left[10 \frac{1}{6} + 0.5 \times \frac{5}{6}\right]
\]
\[
= \rho X_n \cdot \rho = 10 \frac{1}{6} + 0.5 \times \frac{5}{6} \approx 2.1.
\]
Hence,
\[
E[X_{n+1}] = \rho E[X_n], n \geq 1.
\]
Thus,
\[
E[X_n] = \rho^{n-1}, n \geq 1.
\]

Common Mistakes
- $\Omega = \{1,2,3\}$. Define $X,Y$ with $\text{cov}(X,Y) = 0$ and $X,Y$ not independent.
  - Let $X = 0, Y = 1$. No: They are independent.
  - Let $X(1) = -1, X(2) = 0, X(3) = 1, Y(1) = 1, Y(2) = 1, Y(3) = 0$.
  - $3 \times 3.5 = 12.5$. No.
  - $X = B(n,p) \implies \text{var}(X) = np(1-p)$. No.
  - $E[X] = E[X|A] + E[X|\overline{A}]$. No.
  - $\sum_{n=1}^\infty \alpha^n = 1/\alpha$. No.
  - CS70 is difficult. No.
  - I will do poorly on the final. No.
  - Warrand is really weird. Probably!