Lecture Outline

Strengthening Induction Hypothesis.
Strong Induction
Well ordered principle.
How does tutoring work?

1. (Ideally) You work on homework and solve (most of) it.
2. You *do not* need to write-up or turn in.
3. You read and understand homework solutions.
4. You see a tutor, who gives you a short oral quiz.
   4.1 If you do well. Full points.
   4.2 Decent effort. Less than full points.
   4.3 Didn’t understand HW solutions. Uh oh.
   4.4 Can try again. Limit: 2 on average.
Theorem: The sum of the first $n$ odd numbers is a perfect square.

Base Case 1 (0th odd number) is perfect square.

Induction Hypothesis Sum of first $k$ odds is perfect square $a^2 = k^2$.

Induction Step 1. The $(k + 1)$st odd number is $2k + 1$.
2. Sum of the first $k + 1$ odds is $a^2 + 2k + 1 = k^2 + 2k + 1$
3. ??$R^2 + 2k + 1 = (k + 1)^2$

... $P(k+1)!$
Tiling Cory Hall Courtyard.
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer $a$.

\[
2^{2(k+1)} = 2^{2k} \cdot 2^2 \\
= 4 \cdot 2^{2k} \\
= 4 \cdot (3a + 1) \\
= 12a + 3 + 1 \\
= 3(4a + 1) + 1
\]

$a$ integer $\implies (4a + 1)$ is an integer.\qed
**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

What to do now???
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis: 
“Any $2^n \times 2^n$ square can be tiled with a hole *anywhere*.”

Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ... we are done.
Strong Induction.

**Theorem:** Every natural number \( n > 1 \) can be written as a product of primes.

**Fact:** A prime \( n \) has exactly 2 factors 1 and \( n \).

**Base Case:** \( n = 2 \).

**Induction Step:**

\( P(n) = \) “\( n \) can be written as a product of primes.”

Either \( n+1 \) is a prime or \( n+1 = a \cdot b \) where \( 1 < a, b < n+1 \).

\( P(n) \) says nothing about \( a, b! \)

**Strong Induction Principle:** If \( P(0) \) and

\[ (\forall k \in \mathbb{N})( (P(0) \land \ldots \land P(k)) \implies P(k+1)) , \]

then \( (\forall k \in \mathbb{N})(P(k)) \).

\[ P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots \]

Strong induction hypothesis: “\( a \) and \( b \) are products of primes”

\[ \implies “n + 1 = a \cdot b \) is the product of the prime factors” \]

i.e., the product of the factors of \( a \) and the factors of \( b \).
Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \land P(1) \cdots P(k)$.

By the induction principle:
“If $Q(0)$, and $(\forall k \in \mathbb{N})(Q(k) \implies Q(k+1))$ then
$(\forall k \in \mathbb{N})(Q(k))$”

Also, $Q(0) \equiv P(0)$, and $(\forall k \in \mathbb{N})(Q(k)) \equiv (\forall k \in \mathbb{N})(P(k))$

$(\forall k \in \mathbb{N}) (Q(k) \implies Q(k+1))$

$\equiv (\forall k \in \mathbb{N})((P(0) \cdots \land P(k)) \implies (P(0) \cdots P(k) \land P(k+1))$

$\equiv (\forall k \in \mathbb{N})((P(0) \cdots \land P(k)) \implies P(k+1))$

**Strong Induction Principle:** If $P(0)$ and

$(\forall k \in \mathbb{N})((P(0) \land \cdots \land P(k)) \implies P(k+1)),$

then $(\forall k \in \mathbb{N})(P(k))$. 
Well Ordering Principle.

If \( \forall n. P(n) \) is not true, then \( \exists n. \neg P(n) \).

Consider smallest \( m \), with \( \neg P(m) \),
\( P(m - 1) \implies P(m) \) must be false (assuming \( P(0) \) holds.)

This is a proof of the induction principle!
I.e.,
\[
\neg \forall n. P(n) \implies (\exists n, \neg (P(n - 1) \implies P(n))).
\]

(Contrapositive of Induction principle.)

But it assumes that there is a smallest \( m \) where \( P(m) \) does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!
Def: A round robin tournament on $n$ players: each player $p$ plays each player $q$, either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow q$ ($q$ beats $q$.)

Def: A cycle: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.

Theorem: Any tournament that has a cycle has a cycle of length 3.
Tournament has a cycle of length 3 if at all.

Assume the smallest cycle is of length $k$.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.

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p_1 \rightarrow p_3 \Rightarrow 3 \text{ cycle}
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p_3 \rightarrow p_1 \Rightarrow k - 1 \text{ length cycle!}
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Contradiction!
Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ Any $k$ horses have the same color.

Induction step $P(k+1)$?

First $k$ have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

Second $k$ have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$

All $k$ must have the same color. $1, 2, 3, \ldots, k, k+1$

How about $P(1) \implies P(2)$?

Fix base case.

...Still doesn’t work!!

(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn’t work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!
Summary: principle of induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]

Variations:

\[(P(0) \land ((\forall n \in N)(P(n) \implies P(n + 1)))) \implies (\forall n \in N)(P(n))\]

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)

Base Case: Prove \(P(n_0)\).

Ind. Step: Prove. For all values, \(n \geq n_0\), \(P(n) \implies P(n + 1)\).

Statement is proven!

Strong Induction:

\[(P(1) \land ((\forall n \in N)((n \geq 1) \land P(n)) \implies P(n + 1)))) \implies (\forall n \in N)((n \geq 1) \implies P(n))\]