Lecture Outline

Strengthening Induction Hypothesis.
Strong Induction
Well ordered principle.

Tutoring Option.

How does tutoring work?
1. (Ideally) You work on homework and solve (most of) it.
2. You do not need to write-up or turn in.
3. You read and understand homework solutions.
4. You see a tutor, who gives you a short oral quiz.
   4.1 If you do well. 
      Full points.
   4.2 Decent effort. Less than full points.
   4.3 Didn't understand HW solutions. Uh oh.
   4.4 Can try again. Limit: 2 on average.

Strenthening Induction Hypothesis.

Theorem: The sum of the first \( n \) odd numbers is a perfect square.

Theorem: The sum of the first \( n \) odd numbers is \( k^2 \).

Base Case 1 (0th odd number) is perfect square.
Induction Hypothesis: Sum of first \( k \) odds is perfect square \( a^2 = k^2 \).

Induction Step
1. The \((k+1)\)st odd number is \(2k + 1\).
2. Sum of the first \( k+1 \) odds is 
   \[
   a^2 + 2k + 1 = k^2 + 2k + 1
   \]
3. ???
   \[
   k^2 + 2k + 1 = (k+1)^2
   \]

... P(k+1)!

Tiling Cory Hall Courtyard.

B  B
B
C
C
D
D
A
E
E

Hole have to be there? Maybe just one?

Theorem: Any tiling of \( 2^n \times 2^n \) square has to have one hole.

Proof: The remainder of \( 2^{2n} \) divided by 3 is 1.

Base case: true for \( k = 0 \). \( 2^0 = 1 \)
Ind Hyp: \( 2^{2k} = 3a + 1 \) for integer \( a \).

\[
2^{2(k+1)} = 2^{2k} \times 2^2
\]
\[
= 4 \times 2^{2k}
\]
\[
= 4 \times (3a + 1)
\]
\[
= 12a + 3 + 1
\]
\[
= 3(4a + 1) + 1
\]

a integer \( \Rightarrow (4a + 1) \) is an integer.

Hole in center?

Theorem: Can tile the \( 2^n \times 2^n \) square to leave a hole adjacent to the center.

Proof:
Base case: A single tile works fine.
The hole is adjacent to the center of the \( 2 \times 2 \) square.
Induction Hypothesis: Any \( 2^n \times 2^n \) square can be tiled with a hole at the center.

\[
2^{n+1}
\]

What to do now???
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

**Better theorem... better induction hypothesis!**

Base case: Sure. A tile is fine.
Flipping the orientation can leave hole anywhere.

**Induction Hypothesis:**
"Any $2^0 \times 2^0$ square can be tiled with a hole anywhere."

Consider $2^{n-1} \times 2^{n-1}$ square.

Use L-tile and... we are done.

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**Strong Induction.**

**Theorem:** Every natural number $n > 1$ can be written as a product of primes.

**Fact:** A prime $p^n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

$P(n) = "n$ can be written as a product of primes."

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

$P(n)$ says nothing about $a, b$.

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**Strong Induction Principle:** If $P(0)$ and

$(\forall k \in \mathbb{N})((P(0) \land \ldots \land P(k)) \implies P(k + 1))$

then $(\forall k \in \mathbb{N})(P(k))$.

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**Well Ordering Principle.**

If $\forall n.P(n)$ is not true, then $\exists n.\neg P(n)$.

Consider smallest $m$, with $\neg P(m)$,

$P(m−1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!

I.e.,

$\neg \forall n.P(n) \implies (\exists n.\neg P(n−1) \implies P(n))$.

(Contrapositive of Induction principle.)

But it assumes that there is a smallest $m$ where $P(m)$ does not hold.

The Well ordering principle states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principle holds for rationals but with different ordering!!

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**Tournaments have short cycles**

**Def:** A round robin tournament on $n$ players: each player $p$ plays each player $q$, either $p \to q$ ($p$ beats $q$) or $q \to q$ ($q$ beats $q$.)

**Def:** A cycle: a sequence of $p_1, \ldots , p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

**Theorem:** Any tournament that has a cycle has a cycle of length 3.

**Assume the the smallest cycle is of length $k$.**

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.

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**Induction $\implies$ Strong Induction.**

Let $Q(0) = P(0) \land P(1) \land \cdots \land P(0)$.

By the induction principle:

"If $Q(0)$, and $(\forall k \in \mathbb{N})(Q(k) \implies Q(k + 1))$ then $(\forall k \in \mathbb{N})(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in \mathbb{N})(Q(k)) \equiv (\forall k \in \mathbb{N})(P(k))$

$(\forall k \in \mathbb{N})(Q(k) \implies Q(k + 1))$

$\equiv (\forall k \in \mathbb{N})(P(0) \land \cdots \land P(k)) \implies (P(0) \land \cdots \land P(k) \land P(k + 1))$

$\equiv (\forall k \in \mathbb{N})(P(0) \land \cdots \land P(k)) \implies P(k + 1))$

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**Strong Induction Principle:** If $P(0)$ and

$(\forall k \in \mathbb{N})(P(0) \land \cdots \land P(k)) \implies P(k + 1))$, then $(\forall k \in \mathbb{N})(P(k))$.

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**Tournament has a cycle of length 3 if at all.**

Assume the the smallest cycle is of length $k$.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.
**Horses of the same color...**

**Theorem:** All horses have the same color.

**Base Case:** $P(1)$ - trivially true.

**New Base Case:** $P(2)$: there are two horses with same color.

**Induction Hypothesis:** $P(k)$ Any $k$ horses have the same color.

**Induction step $P(k+1)$?**

First $k$ have same color by $P(k)$.
Second $k$ have same color by $P(k)$.
A horse in the middle in common!
All $k$ must have the same color.

How about $P(1) \implies P(2)$?
Fix base case.
...Still doesn’t work!!
(There are two horses is $\neq$ For all two horses!!)
Of course it doesn’t work.
As we will see, it is more subtle to catch errors in proofs of correct theorems!!

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**Summary: principle of induction.**

$$(P(0) \land ((\forall k \in \mathbb{N})(P(k) \implies P(k+1)))) \implies (\forall n \in \mathbb{N})(P(n))$$

Variations:

$$(P(0) \land ((\forall n \in \mathbb{N})(P(n) \implies P(n+1)))) \implies (\forall n \in \mathbb{N})(P(n))$$

Statement to prove: $P(n)$ for $n$ starting from $n_0$

Base Case: Prove $P(n_0)$.

Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.

Statement is proven!

Strong Induction:

$$(P(1) \land ((\forall n \in \mathbb{N})((n \geq 1) \land P(n)) \implies P(n+1)))) \implies (\forall n \in \mathbb{N})((n \geq 1) \implies P(n))$$