Strengthening Induction Hypothesis.
Lecture Outline

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Strong Induction
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Strengthening Induction Hypothesis.
Strong Induction
Well ordered principle.
How does tutoring work?

1. (Ideally) You work on homework and solve (most of) it.
2. You *do not* need to write-up or turn in.
3. You read and understand homework solutions.
4. You see a tutor, who gives you a short oral quiz.
   4.1 If you do well.
   4.2 Decent effort.
   4.3 Didn't understand HW solutions. Uh oh.
   4.4 Can try again. Limit: 2 on average.
5. Begins for second homework.

Questions?
Tutoring Option.

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Questions?
Theorem: The sum of the first \( n \) odd numbers is a perfect square.

**Theorem:** The sum of the first \( n \) odd numbers is \( k^2 \).

- \( k \)th odd number is \( 2(k - 1) + 1 \).

**Base Case** 1 (1th odd number) is \( 1^2 \).

**Induction Hypothesis** Sum of first \( k \) odds is perfect square \( a^2 = k^2 \).

**Induction Step**
1. The \((k + 1)\)st odd number is \( 2k + 1 \).
2. Sum of the first \( k + 1 \) odds is \( a^2 + 2k + 1 = k^2 + 2k + 1 \)
3. \( k^2 + 2k + 1 = (k + 1)^2 \)

... \( P(k+1)! \)
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

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To Tile this $4 \times 4$ courtyard.

A

B

C

D

E
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Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.
Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this $4 \times 4$ courtyard.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles with a center hole.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

**Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.**
with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole)
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole) for every $n$!
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $k = 0$. 

Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer $a$. 
Hole have to be there? Maybe just one?

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$$2^{2(k+1)}$$
Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

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$$2^{2(k+1)} = 2^{2k} \times 2^2$$
Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

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$$2^{2(k+1)} = 2^{2k} \times 2^2$$
$$= 4 \times 2^{2k}$$
Hole have to be there? Maybe just one?

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2^{2(k+1)} = 2^{2k} \times 2^2 \\
= 4 \times 2^{2k} \\
= 4 \times (3a + 1)
\]
Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

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\[
2^{2(k+1)} = 2^{2k} \cdot 2^2 \\
= 4 \cdot 2^{2k} \\
= 4 \cdot (3a + 1) \\
= 12a + 3 + 1
\]
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

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$a$ integer
Hole have to be there? Maybe just one?

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$$2^{2(k+1)} = 2^{2k} \cdot 2^2$$
$$= 4 \cdot 2^{2k}$$
$$= 4 \cdot (3a + 1)$$
$$= 12a + 3 + 1$$
$$= 3(4a + 1) + 1$$

$a$ integer $\implies (4a + 1)$ is an integer.
Hole have to be there? Maybe just one?

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\begin{align*}
2^{2(k+1)} &= 2^{2k} \times 2^2 \\
&= 4 \times 2^{2k} \\
&= 4 \times (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{align*}
\]

$a$ integer $\implies (4a + 1)$ is an integer.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

The hole is adjacent to the center of the $2 \times 2$ square.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
   The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

\[ 2^{n+1} \]

\[
\begin{array}{|c|c|}
\hline
\bullet & \bullet \\
\hline
\bullet & \bullet \\
\hline
\end{array}
\]

\[ 2^n \]

\[ 2^n \]

\[ 2^n \]
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
   The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

What to do now???
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

Better theorem
**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ...better induction hypothesis!
Hole can be anywhere!

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Base case: Sure. A tile is fine.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent _anywhere_.

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole _anywhere_.”

Consider $2^{n+1} \times 2^{n+1}$ square.
Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”
Consider $2^{n+1} \times 2^{n+1}$ square.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole *anywhere.*”
Consider $2^{n+1} \times 2^{n+1}$ square.

```
Use induction hypothesis in each.
```
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ...better induction hypothesis!

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Induction Hypothesis:
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Use induction hypothesis in each.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.” Consider $2^{n+1} \times 2^{n+1}$ square.

Use L-tile and ...
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”
Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ... we are done.
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Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”
Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ... we are done.
Strong Induction.

**Theorem:** Every natural number $n > 1$ can be written as a product of primes.
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**Theorem:** Every natural number $n > 1$ can be written as a product of primes.

**Fact:** A prime $n$ has exactly 2 factors 1 and $n$. 
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Base Case: \( n = 2 \).
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**Induction Step:**
Strong Induction.

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**Induction Step:**

$P(n) =$ “$n$ can be written as a product of primes.”
**Strong Induction.**

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**Fact:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

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$P(n) =$ “$n$ can be written as a product of primes. “

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$. 

Strong Induction.

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**Base Case:** \( n = 2 \).

**Induction Step:**

\( P(n) = \) “\( n \) can be written as a product of primes. “

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

\( P(n) \) says nothing about \( a, b \)!
**Strong Induction.**

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Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

$P(n)$ says nothing about $a, b$!

**Strong Induction Principle:** If $P(0)$ and

$$(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k + 1)),$$

then $(\forall k \in N)(P(k))$. 
**Strong Induction.**

**Theorem:** Every natural number $n > 1$ can be written as a product of primes.

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$$(\forall k \in \mathbb{N})(P(0) \land \ldots \land P(k)) \implies P(k + 1),$$

then $(\forall k \in \mathbb{N})(P(k))$.

$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots$
**Strong Induction.**

**Theorem:** Every natural number $n > 1$ can be written as a product of primes.

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**Strong Induction Principle:** If $P(0)$ and

$$(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k + 1)),$$

then $(\forall k \in N)(P(k))$.

**Strong induction hypothesis:** “$a$ and $b$ are products of primes”
**Strong Induction.**

**Theorem:** Every natural number $n > 1$ can be written as a product of primes.

**Fact:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

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Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

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**Strong Induction Principle:** If $P(0)$ and

$$(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots$

---

Strong induction hypothesis: “$a$ and $b$ are products of primes”

$\implies \text{“} n + 1 = a \cdot b \text{”}$
**Strong Induction.**

**Theorem:** Every natural number \( n > 1 \) can be written as a product of primes.

**Fact:** A prime \( n \) has exactly 2 factors 1 and \( n \).

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**Induction Step:**

\( P(n) = \) “\( n \) can be written as a product of primes. “

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

\( P(n) \) says nothing about \( a, b \)!

**Strong Induction Principle:** If \( P(0) \) and

\[(\forall k \in \mathbb{N})( (P(0) \land \ldots \land P(k)) \implies P(k+1)), \]

then \( (\forall k \in \mathbb{N})(P(k)) \).

\( P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots \)

**Strong induction hypothesis:** “\( a \) and \( b \) are products of primes”

\[ \implies \] “\( n + 1 = a \cdot b = (\text{factorization of } a)(\text{factorization of } b) \)”

\( n + 1 \) can be written as the product of the prime factors!
**Strong Induction.**

**Theorem:** Every natural number \( n > 1 \) can be written as a product of primes.

**Fact:** A prime \( n \) has exactly 2 factors 1 and \( n \).

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P(n) = \text{“} n \text{ can be written as a product of primes. “}\]

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**Strong Induction Principle:** If \( P(0) \) and

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(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k+1)),
\]

then \( (\forall k \in N)(P(k)) \).

\[
P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots
\]

---

**Strong induction hypothesis:** “\( a \) and \( b \) are products of primes”

\[
\implies \text{“} n + 1 = a \cdot b = (\text{factorization of } a)(\text{factorization of } b)\text{”}
\]

\( n + 1 \) can be written as the product of the prime factors!
Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \land P(1) \cdots P(k)$. 
Let $Q(k) = P(0) \land P(1) \cdots P(k)$.

By the induction principle:
“If $Q(0)$, and $(\forall k \in \mathbb{N})(Q(k) \implies Q(k+1))$ then $(\forall k \in \mathbb{N})(Q(k))$”
Induction $\implies$ Strong Induction.

Let $Q(k) = P(0) \land P(1) \cdots P(k)$.

By the induction principle:
“If $Q(0)$, and $(\forall k \in \mathbb{N})(Q(k) \implies Q(k + 1))$ then $(\forall k \in \mathbb{N})(Q(k))$”

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Well Ordering Principle and Induction.

If $(\forall n) P(n)$ is not true, then $(\exists n) \neg P(n)$.

Consider smallest $m$, with $\neg P(m)$, $P(m-1) = \Rightarrow P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle! I.e., $(\neg \forall n) P(n) = \Rightarrow (\exists n) \neg (P(n-1) = \Rightarrow P(n))$.

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest $m$ where $P(m)$ does not hold.

The Well ordering principle states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is "smallest" representation of the representations $a/b$ that represent a single quotient.
Well Ordering Principle and Induction.

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Consider smallest $m$, with $\neg P(m)$,
Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

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$P(m - 1) \implies P(m)$ must be false (assuming $P(0)$ holds.)
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I.e.,

$$(\neg \forall n)P(n) \implies ((\exists n)\neg(P(n - 1) \implies P(n)).$$
Well Ordering Principle and Induction.

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Well Ordering Principle and Induction.

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Tournaments have short cycles

**Def:** A round robin tournament on \( n \) players: every player \( p \) plays every other player \( q \), and either \( p \rightarrow q \) (\( p \) beats \( q \)) or \( q \rightarrow q \) (\( q \) beats \( q \)).
Def: A **round robin tournament on** $n$ **players**: every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow q$ ($q$ beats $q$.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k, p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$. 

**Theorem:** Any tournament that has a cycle has a cycle of length 3.
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Def: A round robin tournament on \( n \) players: every player \( p \) plays every other player \( q \), and either \( p \to q \) (\( p \) beats \( q \)) or \( q \to q \) (\( q \) beats \( q \)).

Def: A cycle: a sequence of \( p_1, \ldots, p_k \), \( p_i \to p_{i+1} \) and \( p_k \to p_1 \).

Theorem: Any tournament that has a cycle has a cycle of length 3.
Tournament has a cycle of length 3 if at all.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.

\[ p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow \cdots \rightarrow p_k \]

\[ p_3 \rightarrow p_1 = \Rightarrow 3 \text{ cycle} \]

Contradiction.

\[ p_1 \rightarrow p_3 = \Rightarrow k - 1 \text{ length cycle!} \]

Contradiction!
Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length $k$.
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```
\begin{array}{l}
\text{"p}_3 \rightarrow \text{p}_1" \quad \Rightarrow \quad \text{3 cycle} \\
\text{Contradiction.}
\end{array}
```
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Assume the smallest cycle is of length $k$.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.

$p_1 \rightarrow p_3 \Rightarrow 3$ cycle

Contradiction.

"$p_3 \rightarrow p_1$" $\Rightarrow$ $k$ – 1 length cycle!

Contradiction!
Horses of the same color...

**Theorem:** All horses have the same color.
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Base Case: $P(1)$ - trivially true.
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Induction Hypothesis: $P(k)$ - Any $k$ horses have the same color.
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Induction step \( P(k + 1) \)?
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First $k$ have same color by $P(k)$. \(1, 2, 3, \ldots, k, k+1\)
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- Second $k$ have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$
- A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$
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- A horse in the middle in common! $1, 2, 3, \ldots, k, k + 1$
- All $k$ must have the same color. $1, 2, 3, \ldots, k, k + 1$
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Second $k$ have same color by $P(k)$.
A horse in the middle in common!

How about $P(1) \implies P(2)$?
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- First $k$ have same color by $P(k)$. $1, 2$
- Second $k$ have same color by $P(k)$. A horse in the middle in common!

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A horse in the middle in common! 1,2
No horse in common!

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Second $k$ have same color by $P(k)$.

A horse in the middle in common!

No horse in common!

How about $P(1) \Rightarrow P(2)$?
Horses of the same color...

**Theorem:** All horses have the same color.

Base Case: $P(1)$ - trivially true.

New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any $k$ horses have the same color.

Induction step $P(k + 1)$?

- First $k$ have same color by $P(k)$.
- Second $k$ have same color by $P(k)$.
  - A horse in the middle in common!

Fix base case.
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...Still doesn’t work!!
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(There are two horses is $\ne$ For all two horses!!!)
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Of course it doesn’t work.
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...Still doesn’t work!!
(There are two horses is \( \neq \) For all two horses!!!)

Of course it doesn’t work.
As we will see, it is more subtle to catch errors in proofs of correct theorems!!
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$. 
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```python
def find_x_y(n):
    if n == 12:
        return (3, 0)
    elif n == 13:
        return (2, 1)
    elif n == 14:
        return (1, 2)
    elif n == 15:
        return (0, 3)
    else:
        (x_prime, y_prime) = find_x_y(n - 4)
        return (x_prime + 1, y_prime)
```

Base cases: $P(12), P(13), P(14), P(15)$.

Strong Induction step: Recursive call is correct: $P(n - 4) \Rightarrow P(n)$.
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Base cases: \( P(12) \)
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        (x’,y’) = find-x-y(n-4)
        return(x’+1,y’)

Base cases: P(12), P(13)
Thm: For every natural number \( n \geq 12 \), \( n = 4x + 5y \).

Instead of proof, let’s write some code!

def find-x-y(n):
    if (n==12): return (3,0)
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Base cases: P(12) , P(13) P(14)
Strong Induction and Recursion.

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Recursive call is correct: \( P(n-4) \)
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Base cases: $P(12), P(13), P(14), P(15)$. Yes.

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Recursive call is correct: $P(n-4) \implies P(n)$. 
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Strong Induction step:

Recursive call is correct: $P(n-4) \implies P(n)$.

Slight differences: showed for all $n \geq 16$ that $\land_{i=4}^{n-1} P(i) \implies P(n)$. 
Summary: principle of induction.

Today: More induction.
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Today: More induction.

\[ P(0) \]
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1))))\]
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]
Summary: principle of induction.

Today: More induction.

\((P(0) \land (((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n)))\)

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \Rightarrow P(k + 1)))) \Rightarrow (\forall n \in N)(P(n))\]

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Base Case: Prove \(P(n_0)\).
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]

Statement to prove: \( P(n) \) for \( n \) starting from \( n_0 \)
Base Case: Prove \( P(n_0) \).
Ind. Step: Prove.
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Base Case: Prove \(P(n_0)\).
Ind. Step: Prove. For all values, \(n \geq n_0\),
Summary: principle of induction.

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Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
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\[ (P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n)) \]

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Statement is proven!
Summary: principle of induction.

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Strong Induction:
Summary: principle of induction.

Today: More induction.

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(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))
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Strong Induction:
\((P(0) \land ((\forall n \in N)(P(n)) \implies P(n+1)))) \implies (\forall n \in N)(P(n))\)

Also Today: strengthened induction hypothesis.
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)

Base Case: Prove \(P(n_0)\).

Ind. Step: Prove. For all values, \(n \geq n_0\), \(P(n) \implies P(n + 1)\).

Statement is proven!

Strong Induction:

\[(P(0) \land ((\forall n \in N)(P(n)) \implies P(n + 1)))) \implies (\forall n \in N)(P(n))\]

Also Today: strengthened induction hypothesis.

**Strengthen theorem statement.**

- Sum of first \(n\) odds is \(n^2\).
- Hole anywhere.

**Not same as strong induction.**
Summary: principle of induction.

Today: More induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]

Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Base Case: Prove \(P(n_0)\).
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Also Today: strengthened induction hypothesis.

**Strengthen theorem statement.**
- Sum of first \(n\) odds is \(n^2\).
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Induction \(\equiv\) Recursion.