Inverses

Today: finding inverses quickly.
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Euclid’s Algorithm.
Inverses

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  Euclid’s Algorithm.
  Runtime.
Inverses

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Euclid’s Algorithm.
Runtime.
Euclid’s Extended Algorithm.
Announcements

From Allen!

1. Starting hw3 checkoff, student will only have two tries at most. The higher grade will be final. Study for the first one!!!

2. OHs and Homework party on Monday and Tuesday, possibly Wednesday are busy with HW checkoff. But usually the early morning (before 12) ones are not crowded, sometimes even empty. Also, Thursday and Friday's are also good choices to avoid the crowds.

3. In order to balance the situation mentioned above, the tutoring deadline is extended to Thursday. Every week but for Thanksgiving. (Something special that week.)
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Does 2 have an inverse mod 8?

Any multiple of 2 is 2 away from 0 + 8k.

Does 2 have an inverse mod 9?

Yes.

5 * 2 = 10 = 1 mod 9.

Does 6 have an inverse mod 9?

No.

Any multiple of 6 is 3 away from 0 + 9k.

x has an inverse modulo m if and only if gcd(x, m) > 1?

No.

gcd(x, m) = 1?

Yes.

Today: Compute gcd!

Compute Inverse modulo m.
Does 2 have an inverse mod 8? No.
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Any multiple of 2 is 2 away from $0 + 8k$. 

Does 2 have an inverse mod 9? Yes.
$5 \times 2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
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$x$ has an inverse modulo $m$ if and only if $\gcd(x, m) > 1$? 
No. $\gcd(x, m) = 1$? Yes. 

Today: Compute gcd! Compute Inverse modulo $m$. 
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    \[ \gcd(x, m) > 1? \text{ No.} \]
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Compute Inverse modulo m.
Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or
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**Fact:** If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$. 
Divisibility...

**Notation:** $d|\,x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d|\,x$ and $d|\,y$ then $d|(x + y)$ and $d|(x - y)$.

**Proof:** $d|\,x$ and $d|\,y$ or
Notation: \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

Fact: If \( d \mid x \) and \( d \mid y \) then \( d \mid (x + y) \) and \( d \mid (x - y) \).

Proof: \( d \mid x \) and \( d \mid y \) or
\[
x = \ell d \quad \text{and} \quad y = kd
\]
Notation: $d| x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

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$x = ℓd$ and $y = kd$

$⇒ x − y = kd − ℓd$
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More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

**Proof:**

\[
\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y = x - s \cdot y
\]

for integer $s = kd - s \ell$ for integers $k, \ell = (k - s \ell)$. Therefore $d|\text{mod}(x, y)$.

And $d|y$ since it is in condition.

**Lemma 2:** If $d|y$ and $d|\text{mod}(x, y)$ then $d|y$ and $d|x$.

**Proof:** Similar.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma. The same common divisors $\Rightarrow$ the largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
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More divisibility

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\text{mod} (x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y \\
= x - [s] \cdot y \quad \text{for integer } s
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$$\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y$$
$$= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s$$
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**Lemma 1:** If \( d | x \) and \( d | y \) then \( d | y \) and \( d | \text{mod} (x, y) \).

**Proof:**

\[
\begin{align*}
\text{mod} (x, y) &= x - \lfloor x/y \rfloor \cdot y \\
&= x - [s] \cdot y \quad \text{for integer } s \\
&= kd - s\ell d \quad \text{for integers } k, \ell \\
&= (k - s\ell)d
\end{align*}
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Therefore \( d | \text{mod} (x, y) \).
More divisibility

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Therefore $d | \text{mod} \ (x, y)$. And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \text{mod} \ (x, y)$ then $d | y$ and $d | x$.

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**GCD Mod Corollary:** $\gcd (x, y) = \gcd (y, \text{mod} \ (x, y))$.

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**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x,y)$.

**Proof:**
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\text{mod}(x,y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y \\
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= (k - s\ell)d
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Therefore $d|\text{mod}(x,y)$. And $d|y$ since it is in condition.

**Lemma 2:** If $d|y$ and $d|\text{mod}(x,y)$ then $d|y$ and $d|x$.

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Therefore $d | \text{mod} (x, y)$. And $d | y$ since it is in condition. \qed

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**Proof...:** Similar. Try this at home.
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**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod}(x, y)$.

**Proof:**

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\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y
$$

$$
= x - \left\lfloor \frac{s}{y} \right\rfloor \cdot y \quad \text{for integer } s
$$

$$
= kd - s\ell d \quad \text{for integers } k, \ell
$$

$$
= (k - s\ell)\ell \quad \text{for integers } k, \ell
$$

Therefore $d \mid \text{mod}(x, y)$. And $d \mid y$ since it is in condition.

**Lemma 2:** If $d \mid y$ and $d \mid \text{mod}(x, y)$ then $d \mid y$ and $d \mid x$.

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$. 
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**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

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**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = k \cdot d \) for some integer \( k \).

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\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer} \ s \\
= k \cdot d - s \ell \cdot d \quad \text{for integers} \ k, \ell \\
= (k - s \ell) \cdot d
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Therefore \( d \mid \text{mod} (x, y) \). And \( d \mid y \) since it is in condition. \( \square \)

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod} (x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar. Try this at home. \( \square \text{ish.} \)

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \ \text{mod} (x, y)) \).

**Proof:** \( x \) and \( y \) have **same** set of common divisors as \( x \) and \( \text{mod} (x, y) \) by Lemma. Same common divisors \( \implies \) largest is the same.
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Same common divisors \( \implies \) largest is the same.
Euclid’s algorithm.

GCD Mod Corollary: \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))) ***
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \ mod \ (x, y))$.

(define (gcd x y)
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**Theorem:** Euclid’s algorithm computes the greatest common divisor of $x$ and $y$ if $x \geq y$. 
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

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(define (gcd x y)
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**Theorem:** Euclid’s algorithm computes the greatest common divisor of \( x \) and \( y \) if \( x \geq y \).

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \text{mod}(x, y)) \).

\[
\text{(define} \ (\text{gcd} \ x \ y) \\
\quad (\text{if} \ (= \ y \ 0) \\
\quad \quad x \\
\quad \quad (\text{gcd} \ y \ (\text{mod} \ x \ y)))) \quad ***
\]

**Theorem:** Euclid’s algorithm computes the greatest common divisor of \( x \) and \( y \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

```
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```

**Theorem:** Euclid’s algorithm computes the greatest common divisor of $x$ and $y$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”

$\implies$ “$x$ is common divisor and clearly largest.”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).

\[
\text{(define (gcd x y)}
  \begin{align*}
    & (\text{if (= y 0)} \\
    & \quad x \\
    & \quad (\text{gcd y (mod x y)})))) \quad ***
  \end{align*}
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**Theorem:** Euclid’s algorithm computes the greatest common divisor of \( x \) and \( y \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

```
(define (gcd x y)
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**Theorem:** Euclid’s algorithm computes the greatest common divisor of $x$ and $y$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”

$$\implies \text{“}x\text{ is common divisor and clearly largest.”}$$

**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$

Call in line (***)) meets conditions plus arguments “smaller”
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

```
(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y))))
```

**Theorem:** Euclid’s algorithm computes the greatest common divisor of $x$ and $y$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”

$\implies$ “$x$ is common divisor and clearly largest.”

**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$

Call in line (***') meets conditions plus arguments “smaller” and by strong induction hypothesis
Euclid’s algorithm.

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Call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes $\gcd(x, \mod(x, y))$
Euclid’s algorithm.

**GCD Mod Corollary:** $\text{gcd}(x, y) = \text{gcd}(y, \ mod \ (x, y))$.

```
(define (gcd x y)
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```

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The call in line (***)) meets conditions plus arguments “smaller”

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computes $\text{gcd}(x, \ \text{mod} \ (x, y))$

which is $\text{gcd}(x, y)$ by GCD Mod Corollary.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

```scheme
(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y))))
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**Theorem:** Euclid’s algorithm computes the greatest common divisor of \( x \) and \( y \) if \( x \geq y \).

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Call in line (***) meets conditions plus arguments “smaller” and by strong induction hypothesis computes \( \gcd(x, \mod(x, y)) \) which is \( \gcd(x, y) \) by GCD Mod Corollary.
Excursion: Value and Size.

Before discussing running time of gcd procedure...
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
Before discussing running time of gcd procedure...

What is the value of 1,000,000?
one million or 1,000,000!

What is the “size” of 1,000,000?

Number of digits: 7.
Number of bits: 21.

For a number $x$, what is its size in bits?

$n = b(x) \approx \log_2 x$
Excursion: Value and Size.

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$$n = b(x) \approx \log_2 x$$
GCD procedure is fast.

**Theorem**: GCD uses $2n$ ”divisions” where $n$ is the number of bits.
GCD procedure is fast.

**Theorem:** GCD uses $2n$ "divisions" where $n$ is the number of bits.
Is this good?
GCD procedure is fast.

**Theorem:** GCD uses $2n$ "divisions" where $n$ is the number of bits. Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?
GCD procedure is fast.

**Theorem:** GCD uses $2n$ “divisions” where $n$ is the number of bits.
Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?
Check 2,
Theorem: GCD uses $2n$ ”divisions” where $n$ is the number of bits.
Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?
Check 2, check 3,
GCD procedure is fast.

**Theorem:** GCD uses $2n$ ”divisions” where $n$ is the number of bits.

Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?

Check 2, check 3, check 4,
GCD procedure is fast.

**Theorem:** GCD uses $2n$ "divisions" where $n$ is the number of bits. Is this good? Better than trying all numbers in \{2, \ldots, y/2\}? Check 2, check 3, check 4, check 5 \ldots, check $y/2$. 

$2^n - 1$ divisions! Exponential dependence on size!

A 101-bit number.

$2^{100} \approx 10^{30} = \text{million, trillion, trillion}$ divisions!

$2^n$ is much faster! Roughly 200 divisions.
GCD procedure is fast.

Theorem: GCD uses $2n$ "divisions" where $n$ is the number of bits. Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$? Check 2, check 3, check 4, check 5 \ldots, check $y/2$.

$2^{n-1}$ divisions! Exponential dependence on size!
GCD procedure is fast.

**Theorem:** GCD uses $2n$ ”divisions” where $n$ is the number of bits.

Is this good? Better than trying all numbers in \{2, \ldots , y/2\}?

Check 2, check 3, check 4, check 5 \ldots , check $y/2$.

$2^{n-1}$ divisions! Exponential dependence on size!

101 bit number.
Theorem: GCD uses $2n$ ”divisions” where $n$ is the number of bits.

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$2^{n-1}$ divisions! Exponential dependence on size!
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Algorithms at work.

Trying everything

Try different things.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$. 

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
(The second is less than the first.)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
gcd(700, 568)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 \ldots, check \( y/2 \).

\((\gcd x y)\) at work.

\[
\begin{align*}
gcd(700, 568) \\
gcd(568, 132)
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\[
\begin{align*}
gcd(700, 568) \\
gcd(568, 132) \\
gcd(132, 40)
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

```
gcd(700, 568)
gcd(568, 132)
gcd(132, 40)
gcd(40, 12)
```
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\[
\begin{align*}
gcd(700, 568) \\
gcd(568, 132) \\
gcd(132, 40) \\
gcd(40, 12) \\
gcd(12, 4) 
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd $x$ $y$)” at work.

\[
gcd(700, 568) \\
gcd(568, 132) \\
gcd(132, 40) \\
gcd(40, 12) \\
gcd(12, 4) \\
gcd(4, 0)
\]
Algorithms at work.

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.
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\begin{align*}
\text{gcd}(700, 568) \\
\text{gcd}(568, 132) \\
\text{gcd}(132, 40) \\
\text{gcd}(40, 12) \\
\text{gcd}(12, 4) \\
\text{gcd}(4, 0) \\
4
\end{align*}
\]
Algorithms at work.

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.
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Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
Algorithms at work.

Trying everything
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```
gcd(700, 568)
gcd(568, 132)
gcd(132, 40)
gcd(40, 12)
gcd(12, 4)
gcd(4, 0)
  4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
(The second is less than the first.)
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y))))

**Theorem:** GCD uses $O(n)$ "divisions" where $n$ is the number of bits.
Proof.

\[
\begin{align*}
\text{(define (gcd x y)} & \\
\text{  (if (= y 0)} & \text{ x)} & \\
\text{  ) (gcd y (mod x y))})
\end{align*}
\]

**Theorem:** GCD uses $O(n)$ ”divisions” where $n$ is the number of bits.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

**Theorem:** GCD uses $O(n)$ "divisions" where $n$ is the number of bits.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
Proof.

(define (gcd x y)
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      x
      (gcd y (mod x y)))))

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After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number. One more recursive call to finish.
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After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
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1 division per recursive call.
$O(n)$ divisions.

\[\square\]
(define (gcd x y)
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**Theorem:** GCD uses $O(n)$ "divisions" where $n$ is the number of bits.

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**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.
Proof.

(define (gcd x y)
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**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Theorem: GCD uses $O(n)$ ”divisions” where $n$ is the number of bits.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2$.”
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Theorem: GCD uses $O(n)$ "divisions" where $n$ is the number of bits.
Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.
Proof of Fact: Recall that first argument decreases every call.
Case 2: Will show "$y \geq x/2$" $\implies$ "mod($x, y) \leq x/2.""
    mod ($x, y$) is second argument in next recursive call,
Proof.

\[
\text{(define (gcd x y)}
\text{  (if (= y 0)}
\text{    x)}
\text{  (gcd y (mod x y)))}
\]

**Theorem:** GCD uses \( O(n) \) ”divisions” where \( n \) is the number of bits.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

Case 2: Will show “\( y \geq x/2 \) \( \implies \) “mod(x, y) \( \leq x/2.\)”

\[
\text{mod (x, y) is second argument in next recursive call,}
\text{ and becomes the first argument in the next one.}
\]
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Theorem: GCD uses $O(n)$ ”divisions” where $n$ is the number of bits.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 2: Will show “$y \geq x/2$” $\implies$ "mod($x, y$) $\leq x/2$.”

When $y \geq x/2$, then

$$\left\lfloor \frac{x}{y} \right\rfloor = 1,$$
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Theorem: GCD uses $O(n)$ "divisions" where $n$ is the number of bits.
Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.
Proof of Fact: Recall that first argument decreases every call.
Case 2: Will show \( y \geq x/2 \implies \text{mod}(x, y) \leq x/2. \)
When $y \geq x/2$, then
\[
\left\lfloor \frac{x}{y} \right\rfloor = 1,
\]
\[
\text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor =
\]
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Theorem: GCD uses $O(n)$ ”divisions” where $n$ is the number of bits.

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First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2.$”
When $y \geq x/2$, then

$$\left\lfloor \frac{x}{y} \right\rfloor = 1,$$

$$\text{mod} \,(x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2$$
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y))))

Theorem: GCD uses $O(n)$ ”divisions” where $n$ is the number of bits.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2.$”
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$$\text{mod}(x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2 = x/2$$
Proof.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Theorem: GCD uses $O(n)$ ”divisions” where $n$ is the number of bits.

Proof:

Fact: First arg decreases by at least factor of two in two recursive calls. After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number. One more recursive call to finish. $O(n)$ divisions.

Proof of Fact: Recall that first argument decreases every call.

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2.$” When $y \geq x/2$, then

$$\left\lfloor \frac{x}{y} \right\rfloor = 1,$$

$$\text{mod} \ (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2 = x/2$$
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Finding an inverse?

We showed how to efficiently tell if there is an inverse. Extend euclid to find inverse.
Euclid’s GCD algorithm.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y))))
Euclid’s GCD algorithm.

(define (gcd x y)
  (if (= y 0)
      x
      (gcd y (mod x y)))))

Computes the gcd(x, y) in $O(n)$ divisions.
Euclid’s GCD algorithm.

\[
\text{(define (gcd x y)}
\begin{align*}
&\text{(if (= y 0)} \\
&\quad x \\
&\quad (gcd \ y \ (mod \ x \ y)))
\end{align*}
\]

Computes the \( \text{gcd}(x, y) \) in \( O(n) \) divisions.

For \( x \) and \( m \), if \( \text{gcd}(x, m) = 1 \) then \( x \) has an inverse modulo \( m \).
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse.
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse. How do we find a multiplicative inverse?
Extended GCD

**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that
\[ ax + by = \gcd(x, y) \]
Extended GCD

**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that

$$ax + by = \gcd(x, y) = d \quad \text{where } d = \gcd(x, y).$$
Extended GCD

**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that

$$ax + by = \gcd(x, y) = d$$

where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”
Extended GCD

Euclid’s Extended GCD Theorem: For any \( x, y \) there are integers \( a, b \) such that
\[
ax + by = \gcd(x, y) = d \quad \text{where} \quad d = \gcd(x, y).
\]

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
$$ax + by = \gcd(x, y) = d \quad \text{where } d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$. 
Extended GCD

**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that

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By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = \gcd(x, y) = d \quad \text{where } d = \gcd(x, y).$$

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What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$
**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that
\[ ax + by = \gcd(x, y) = d \quad \text{where} \quad d = \gcd(x, y). \]

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

\[
ax + bm = 1 \\
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So $a$ multiplicative inverse of $x$ if $\gcd(a, x) = 1$!!
**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that

$$ax + by = \gcd(x, y) = d \quad \text{where } d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

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By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$ 

So a multiplicative inverse of $x$ if $\gcd(a, x) = 1$!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$. 

$$3 \times 12 + (-1) \times 35 = 1.$$ 

$a = 3$ and $b = -1$. The multiplicative inverse of 12 (mod 35) is 3.
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = \gcd(x, y) = d \quad \text{where} \quad d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$ 

So $a$ multiplicative inverse of $x$ if $\gcd(a, x) = 1$!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = \gcd(x, y) = d$$

where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$ 

So $a$ multiplicative inverse of $x$ if $\gcd(a, x) = 1$!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$ 

$a = 3$ and $b = -1$. 
**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that

$$ax + by = \text{gcd}(x, y) = d$$

where $d = \text{gcd}(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\text{gcd}(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$  

So $a$ multiplicative inverse of $x$ if $\text{gcd}(a, x) = 1$!!

**Example:** For $x = 12$ and $y = 35$, $\text{gcd}(12, 35) = 1$.

$$(3)12 + (−1)35 = 1.$$  

$a = 3$ and $b = −1$.

The multiplicative inverse of $12 \pmod{35}$ is $3$.  

Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12)
\]

How did $gcd$ get 11 from 35 and 12?

\[
35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2 \times 12) = 11
\]

How does $gcd$ get 1 from 12 and 11?

\[
12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - (1 \times 11) = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

1 = 12 - (1 \times 11) = 12 - (1 \times (35 - (2 \times 12))) = 3 \times 12 + (-1) \times 35

Get 11 from 35 and 12 and plugin....

Simplify.

\[
a = 3 \text{ and } b = -1.
\]
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, 35\%12)
\]
Make $d$ out of $x$ and $y$..?

$$\text{gcd}(35, 12)$$
$$\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \% 12)$$
$$\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \% 11)$$
Make $d$ out of $x$ and $y$..?

$$\text{gcd}(35, 12)$$
$$\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \% 12)$$
$$\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \% 11)$$
$$\text{gcd}(1, 0)$$
$$1$$

How did gcd get 11 from 35 and 12?

$$35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2 \times 12) = 11$$

How does gcd get 1 from 12 and 11?

$$12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - (1 \times 11) = 1$$

The algorithm finally returns 1.

But we want 1 from the sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$$1 = 12 - (1 \times 11) = 12 - (1 \times (35 - (2 \times 12))) = (3 \times 12) + (-1) 	imes 35$$

Get 11 from 35 and 12 and plugin...

Simplify.

$$a = 3 \text{ and } b = -1.$$
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, \ 35\%12) \\
gcd(11, 1) ;; gcd(11, \ 12\%11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
Make \( d \) out of \( x \) and \( y \)...

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \mod 12) \\
gcd(11, 1) ;; gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor \cdot 12 = 35 - (2)12 = 11
\]
Make $d$ out of $x$ and $y$..?

\[\gcd(35,12)\]
\[\gcd(12, 11) \;;\; \gcd(12, 35 \mod 12)\]
\[\gcd(11, 1) \;;\; \gcd(11, 12 \mod 11)\]
\[\gcd(1,0)\]
\[1\]

How did $\gcd$ get 11 from 35 and 12?
\[35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2)12 = 11\]

How does $\gcd$ get 1 from 12 and 11?
Make $d$ out of $x$ and $y$..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, 35 \% 12) \\
gcd(11, 1) ;; \gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did $gcd$ get 11 from 35 and 12?

\[
35 - \left\lfloor \frac{35}{12} \right\rfloor \cdot 12 = 35 - (2)12 = 11
\]

How does $gcd$ get 1 from 12 and 11?

\[
12 - \left\lfloor \frac{12}{11} \right\rfloor \cdot 11 = 12 - (1)11 = 1
\]
Make $d$ out of $x$ and $y$..?

\[
gcd(35,12) \\
gcd(12, 11) ;; gcd(12, 35\%12) \\
gcd(11, 1) ;; gcd(11, 12\%11) \\
gcd(1,0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.
Make \( d \) out of \( x \) and \( y \) ..?

\[
\begin{align*}
gcd(35,12) \\
gcd(12, 11) & \quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) & \quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1,0) & \\
1
\end{align*}
\]

How did \( gcd \) get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor \cdot 12 = 35 - (2)12 = 11
\]

How does \( gcd \) get 1 from 12 and 11?
\[
12 - \lfloor \frac{12}{11} \rfloor \cdot 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Make \( d \) out of \( x \) and \( y \)...

\[
\text{gcd}(35, 12)
\]
\[
\text{gcd}(12, 11) ;; \quad \text{gcd}(12, 35 \mod 12)
\]
\[
\text{gcd}(11, 1) ;; \quad \text{gcd}(11, 12 \mod 11)
\]
\[
\text{gcd}(1, 0)
\]
\[
1
\]

How did \( \text{gcd} \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \text{gcd} \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12) \\
gcd(11, 1) ;; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor\frac{35}{12}\right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor\frac{12}{11}\right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11
\]
Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( gcd \) get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11
\]

How does \( gcd \) get 1 from 12 and 11?
\[
12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12)
\]

Get 11 from 35 and 12 and plugin....
Make \( d \) out of \( x \) and \( y \)..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) & ; ; \gcd(12, 35 \div 12) \\
gcd(11, 1) & ; ; \gcd(11, 12 \div 11) \\
gcd(1, 0) & \\
1
\end{align*}
\]

How did \( \gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify.
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) \;; \; gcd(12, 35 \% 12) \\
gcd(11, 1) \;; \; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11\]

How does gcd get 1 from 12 and 11?
\[12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (\color{red}{-1})35\]

Get 11 from 35 and 12 and plugin....  Simplify.
Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did gcd get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify. \( a = 3 \) and \( b = -1 \).
Extended GCD Algorithm.

\[ \text{ext-gcd}(x, y) \]
\[
\begin{align*}
\text{if } y &= 0 \text{ then return } (x, 1, 0) \\
\text{else} & \\
(d, a, b) & := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]
\[
\text{else}
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Extended GCD Algorithm.

```
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)
```

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

```
ext-gcd(35, 12)
```
Extended GCD Algorithm.

```
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)
```

Claim: Returns \((d, a, b)\): \(d = \text{gcd}(a, b)\) and \(d = ax + by\).

Example:

```
ext-gcd(35, 12)
ext-gcd(12, 11)
```
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

if \( y = 0 \) then return \((x, 1, 0)\)
else
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y))
\]
return \((d, b, a - \text{floor}(x/y) \times b)\)

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]
\[
\text{ext-gcd}(12, 11)
\]
\[
\text{ext-gcd}(11, 1)
\]
Extended GCD Algorithm.

```
ext-gcd(x, y)
  if y = 0 then return(x, 1, 0)
  else
    (d, a, b) := ext-gcd(y, mod(x, y))
    return (d, b, a - floor(x/y) * b)
```

Claim: Returns $(d, a, b)$: $d = gcd(a, b)$ and $d = ax + by$.

Example:

```
ext-gcd(35, 12)
ext-gcd(12, 11)
ext-gcd(11, 1)
ext-gcd(1, 0)
```
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\begin{align*}
\text{if } y &= 0 \text{ then return }(x, 1, 0) \\
\text{else} & \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\text{return } (d, b, a - \text{floor}(x/y) * b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - [x/y] \cdot b = \)

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
\text{ext-gcd}(11, 1) \\
\text{ext-gcd}(1, 0) \\
\text{return } (1, 1, 0) ;; 1 = (1)1 + (0) 0
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - [11/1] \cdot 0 = 1\)

\[
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
\text{ext-gcd}(11, 1) \\
\text{ext-gcd}(1, 0) \\
\text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0 \\
\text{return } (1, 0, 1) ;; 1 = (0)11 + (1)1
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x,y) \\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \cdot b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a,b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1\)

\[
\text{ext-gcd}(35,12) \\
\text{ext-gcd}(12, 11) \\
\text{ext-gcd}(11, 1) \\
\text{ext-gcd}(1,0) \\
\text{return } (1,1,0) ;; 1 = (1)1 + (0) 0 \\
\text{return } (1,0,1) ;; 1 = (0)11 + (1)1 \\
\text{return } (1,1,-1) ;; 1 = (1)12 + (-1)11
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example: \(a - \lfloor x/y \rfloor \cdot b = \lfloor 35/12 \rfloor \cdot (-1) = 3\)

\[
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
\text{ext-gcd}(11, 1) \\
\text{ext-gcd}(1, 0) \\
\quad \text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0 \\
\quad \text{return } (1, 0, 1) ;; 1 = (0)11 + (1)1 \\
\quad \text{return } (1, 1, -1) ;; 1 = (1)12 + (-1)11 \\
\quad \text{return } (1, -1, 3) ;; 1 = (-1)35 + (3)12
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\begin{align*}
\quad & \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad & \text{else} \\
\quad & \quad (d, a, b) := \text{ext-gcd}(y, \mod(x, y)) \\
\quad & \quad \text{return } (d, b, a - \text{floor}(x/y) \ast b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\quad \text{ext-gcd}(12, 11) \\
\quad \text{ext-gcd}(11, 1) \\
\quad \text{ext-gcd}(1, 0) \\
\quad \quad \text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0 \\
\quad \quad \text{return } (1, 0, 1) ;; 1 = (0)11 + (1)1 \\
\quad \quad \text{return } (1, 1, -1) ;; 1 = (1)12 + (-1)11 \\
\quad \quad \text{return } (1, -1, 3) ;; 1 = (-1)35 + (3)12
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\qquad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\qquad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]
Extended GCD Algorithm.

```
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)
```

**Theorem:** Returns \((d, a, b)\), where \(d = gcd(a, b)\) and

\[ d = ax + by. \]
Correctness.

Proof: Strong Induction.¹

¹Assume $d$ is $\text{gcd}(x, y)$ by previous proof.
Correctness.

Proof: Strong Induction. \(^1\)

Base: ext-gcd(x, 0) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** ext-gcd\((x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: ext-gcd\((y, \text{mod}(x, y))\) returns \((d, a, b)\) with 
\[
d = ay + b(\text{mod}(x, y))
\]
Correctness.

**Proof:** Strong Induction.  
**Base:** \( \text{ext-gcd}(x, 0) \) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)  
Ind hyp: \( \text{ext-gcd}(y, \mod(x, y)) \) returns \((d, a, b)\) with  
\[
d = ay + b(\mod(x, y))
\]
\(\text{ext-gcd}(x, y)\) calls \( \text{ext-gcd}(y, \mod(x, y)) \) so  
\[
d = ay + b(\mod(x, y)) = bx + (a - \lfloor x/y \rfloor \cdot b)y
\]

\[\text{Assume } d \text{ is } \gcd(x, y) \text{ by previous proof.}\]
Correctness.

Proof: Strong Induction.¹
Base: ext-gcd(x, 0) returns (d = x, 1, 0) with x = (1)x + (0)y.

Induction Step: Returns (d, A, B) with d = Ax + By
Ind hyp: ext-gcd(y, mod(x, y)) returns (d, a, b) with
\[ d = ay + b \mod(x, y) \]

ext-gcd(x, y) calls ext-gcd(y, mod(x, y)) so
\[ d = ay + b \cdot ( \mod(x, y) ) \]

¹Assume d is gcd(x, y) by previous proof.
Correctness.

**Proof:** Strong Induction.¹

**Base:** `ext-gcd(x, 0)` returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: `ext-gcd(y, mod(x, y))` returns \((d, a, b)\) with \(d = ay + b(\mod(x, y))\)

`ext-gcd(x, y)` calls `ext-gcd(y, mod(x, y))` so

\[
\begin{align*}
    d &= ay + b(\mod(x, y)) \\
    &= ay + b(x - \lfloor \frac{x}{y} \rfloor y)
\end{align*}
\]

¹Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.¹
Base: ext-gcd(x, 0) returns (d = x, 1, 0) with x = (1)x + (0)y.

Induction Step: Returns (d, A, B) with d = Ax + By
Ind hyp: ext-gcd(y, mod(x, y)) returns (d, a, b) with d = ay + b(mod(x, y))

ext-gcd(x, y) calls ext-gcd(y, mod(x, y)) so

\[ d = ay + b \cdot (\text{mod}(x, y)) \]
\[ = ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y) \]
\[ = bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y \]

¹Assume d is gcd(x, y) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with \(d = ay + b(\mod(x, y))\)

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

\[
d = ay + b \cdot (\mod(x, y))
\]

\[
= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y)
\]

\[
= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\]

And \(\text{ext-gcd}\) returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\) so theorem holds!

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with \(d = ay + b\mod(x, y))\)

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

\[
\begin{align*}
d & = ay + b \cdot (\mod(x, y)) \\
& = ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\
& = bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\end{align*}
\]

And \(\text{ext-gcd}\) returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\) so theorem holds! \(\square\)

\(^1\) Assume \(d\) is \(gcd(x, y)\) by previous proof.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \cdot b)
\]

Recursively:
\[
d = ay + bx - (a - \text{floor}(x/y) \cdot b)y
\]
Returns \((d, b, a - \text{floor}(x/y) \cdot b))\).

```
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

Recursively: \( d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \)
```
ext-gcd(x, y)
if y = 0 then return(x, 1, 0)
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    (d, a, b) := ext-gcd(y, mod(x, y))
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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

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\text{return } (d, b, a - \text{floor}(x/y) \cdot b)
\]

Recursively: \( d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y \)

Returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\).
Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!
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Very different from elementary school: try 1, try 2, try 3...
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Inverse of 500,000,357 modulo 1,000,000,000,000,000?
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Inverse of $500,000,357$ modulo $1,000,000,000,000,000$? $\leq 80$ divisions.
Wrap-up

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versus 1,000,000
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Inverse of 500,000,357 modulo 1,000,000,000,000,000? $\leq 80$ divisions.
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Internet Security.
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Public Key Cryptography: 512 digits.
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Next Week!