Today.

Polynomials.
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Polynomials.
Secret Sharing.
A secret!

I have a secret!
I have a secret!
A number from 0 to 10.
A secret!

I have a secret!
A number from 0 to 10.
What is it?
A secret!

I have a secret!
A number from 0 to 10.
What is it?
   Any one of you knows nothing!
I have a secret!

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What is it?

   Any one of you knows nothing!
   Any two of you can figure it out!
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Example Applications:
A secret!

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Example Applications:
  Nuclear launch: need at least 3 out of 5 people to launch!
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Example Applications:
Nuclear launch: need at least 3 out of 5 people to launch!
Cloud service backup: several vendors, each knows nothing.
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Example Applications:
Nuclear launch: need at least 3 out of 5 people to launch!
Cloud service backup: several vendors, each knows nothing.
   data from any 2 to recover data.
Secret Sharing.

Share secret among $n$ people.

Secrecy: Any $k-1$ knows nothing.

Robustness: Any $k$ knows secret.

Efficient: minimize storage.
Secret Sharing.

Share secret among $n$ people.
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Polynomials

A polynomial

\[ P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0. \]

is specified by **coefficients** \( a_d, \ldots a_0 \).

\(^1\) A field is a set of elements with addition and multiplication operations, with inverses. \( GF(p) = (\{0, \ldots, p-1\}, + \text{ (mod } p) , \star \text{ (mod } p)) \).
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\( P(x) \) contains point \((a, b)\) if \( b = P(a) \).
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Polynomials over reals: \( a_1, \ldots , a_d \in \mathbb{R} \), use \( x \in \mathbb{R} \).

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Polynomials over reals: \( a_1, \ldots, a_d \in \mathbb{R} \), use \( x \in \mathbb{R} \).

Polynomials \( P(x) \) with arithmetic modulo \( p \): \(^1\) \( a_i \in \{0, \ldots, p-1\} \) and

\[ P(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_0 \pmod{p}, \]

for \( x \in \{0, \ldots, p-1\} \).

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Polynomial: $P(x) = a_d x^4 + \cdots + a_0$

Line: $P(x) = a_1 x + a_0$
Polynomial: $P(x) = a_dx^4 + \cdots + a_0$

Line: $P(x) = a_1x + a_0 = mx + b$
Polynomial: \( P(x) = a_d x^4 + \cdots + a_0 \)

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\[ P(x) \]

\[ \rightarrow x \]
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Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$
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Polynomial: $P(x) = a_d x^4 + \cdots + a_0 \pmod{p}$

Finding an intersection.

$x + 2 \equiv 3x + 1 \pmod{5} \\ \Rightarrow 2x \equiv 1 \pmod{5} \\ \Rightarrow x \equiv 3 \pmod{5}$

3 is multiplicative inverse of 2 modulo 5.

Good when modulus is prime!!
Polynomial: \( P(x) = a_d x^4 + \cdots + a_0 \pmod p \)

Finding an intersection.

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3 is multiplicative inverse of 2 modulo 5.
Good when modulus is prime!!
Two points make a line.

**Fact:** Exactly 1 degree \( \leq d \) polynomial contains \( d + 1 \) points. \(^2\)

\(^2\)Points with different \( x \) values.
Two points make a line.

**Fact:** Exactly 1 degree $\leq d$ polynomial contains $d + 1$ points. $^2$

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Two points specify a line. $d = 1$,

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Two points specify a line. $d = 1$, $1 + 1$ is 2!

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Three points specify a parabola.

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Three points specify a parabola. $d = 2$, $2 + 1 = 3$.

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Three points specify a parabola. $d = 2$, $2 + 1 = 3$.

**Modular Arithmetic Fact:** Exactly $1$ degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d + 1$ pts.

---

\(^2\)Points with different $x$ values.
3 points determine a parabola.

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$P(x) = 0.5x^2 - x + 1$
3 points determine a parabola.

Fact: Exactly 1 degree \( \leq d \) polynomial contains \( d + 1 \) points.
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Fact: Exactly 1 degree \( \leq d \) polynomial contains \( d + 1 \) points. \(^3\)

\(^3\)Points with different \( x \) values.
2 points not enough.

There is $P(x)$ contains blue points and any $(0, y)$!
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\[
P(x) = 0.2x^2 - 0.5x + 1.5
\]

\[
P(x) = -0.3x^2 + 1x + 0.5
\]

\[
P(x) = -0.6x^2 + 1.9x - 0.1
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2 points not enough.

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Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact:
Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d + 1$ pts.

Shamir's $k$ out of $n$ Scheme:
Secrets $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and randomly $a_1, \ldots, a_{k-1}$.

2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0$ with $a_0 = s$.

3. Share $i$ is point $(i, P(i) \mod p)$.

Roubustness: Any $k$ shares gives secret.

Knowing $k$ pts $\Rightarrow$ only one $P(x)$ $\Rightarrow$ evaluate $P(0)$.

Secrecy: Any $k-1$ shares give nothing.

Knowing $\leq k-1$ pts $\Rightarrow$ any $P(0)$ is possible.
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Knowing $k$ pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k - 1$ shares give nothing.
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What’s my secret?

Remember:
Secret: number from 0 to 10.
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Shares: points on a line.
What’s my secret?

Remember:
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Shares: points on a line.
Secret: $y$-intercept.
What’s my secret?

Remember:
Secret: number from 0 to 10.
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Shares: points on a line.
Secret: $y$-intercept.
Arithmetic Modulo 11.
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Arithmetic Modulo 11.
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Arithmetic Modulo 11.

What’s my secret?
From \( d + 1 \) points to degree \( d \) polynomial?

For a line, \( a_1 x + a_0 = mx + b \) contains points \((1, 3)\) and \((2, 4)\).
From \(d + 1\) points to degree \(d\) polynomial?

For a line, \(a_1 x + a_0 = mx + b\) contains points \((1,3)\) and \((2,4)\).

\[
P(1) =
\]
From $d + 1$ points to degree $d$ polynomial?

For a line, $a_1 x + a_0 = mx + b$ contains points $(1, 3)$ and $(2, 4)$.

$$P(1) = m(1) + b \equiv m + b$$
From $d+1$ points to degree $d$ polynomial?

For a line, $a_1 x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

\[ P(1) = m(1) + b \equiv m + b \equiv 3 \pmod{5} \]
From $d + 1$ points to degree $d$ polynomial?

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\[
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Subtract first from second.

$$m + b \equiv 3 \pmod{5}$$
$$m \equiv 1 \pmod{5}$$

Backsolve:

$$b \equiv 2 \pmod{5}$$

Secret is 2.

And the line is...

$$x + 2 \mod 5$$
From $d + 1$ points to degree $d$ polynomial?

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Subtract first from second..

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m + b \equiv 3 \pmod{5}
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Backsolve: $b \equiv 2 \pmod{5}$.

Secret is 2.

And the line is $x + 2 \pmod{5}$. 
From $d + 1$ points to degree $d$ polynomial?

For a line, $a_1 x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

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\begin{align*}
P(1) &= m(1) + b \\ &\equiv m + b \equiv 3 \pmod{5} \\ P(2) &= m(2) + b \\ &\equiv 2m + b \equiv 4 \pmod{5}
\end{align*}
\]

Subtract first from second..

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\begin{align*}
m + b &\equiv 3 \pmod{5} \\
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Backsolve: \( b \equiv 2 \pmod{5} \). **Secret is 2.**

And the line is...

\[
x + 2 \pmod{5}.
\]
What’s my secret?

\[ P(1) = m(1) + b \equiv 5 \pmod{11} \]
\[ P(3) = m(3) + b \equiv 9 \pmod{11} \]

Subtract first from second.
\[ 2m \equiv 4 \pmod{11} \]

Multiplicative inverse of 2 \((\mod 11)\) is 6:
\[ 6 \times 2 \equiv 12 \equiv 1 \pmod{11} \]

Multiply both sides by 6.
\[ 12m \equiv 24 \pmod{11} \]
\[ m \equiv 2 \pmod{11} \]

Backsolve: 2 + b \equiv 5 \pmod{11}.

Or \[ b \equiv 3 \pmod{11} \].

Secret is 3.
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Secret is 3.
Quadratic

For a quadratic polynomial, \( a_2 x^2 + a_1 x + a_0 \) hits \((1, 2); (2, 4); (3, 0)\).
For a quadratic polynomial, \( a_2x^2 + a_1x + a_0 \) hits (1, 2); (2, 4); (3, 0). Plug in points to find equations.
Quadratic

For a quadratic polynomial, $a_2 x^2 + a_1 x + a_0$ hits $(1,2); (2,4); (3,0)$. Plug in points to find equations.

\[ P(1) = a_2 + a_1 + a_0 \equiv 2 \pmod{5} \]
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Subtracting 2nd from 3rd yields: \( a_1 = 1 \).
For a quadratic polynomial, $a_2 x^2 + a_1 x + a_0$ hits (1,2); (2,4); (3,0).
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a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1})
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So polynomial is \(2x^2 + 1x + 4 \pmod{5}\)
In general: Linear System.

Given points: \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).
In general: Linear System.

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Solve...

\[
a_{k-1}x_1^{k-1} \cdots + a_0 \equiv y_1 \pmod{p}
\]

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a_{k-1}x_2^{k-1} \cdots + a_0 \equiv y_2 \pmod{p}
\]

\[
\cdots
\]

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Will this always work?
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As long as solution \textbf{exists} and it is \textbf{unique}! And...
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**Modular Arithmetic Fact:** Exactly 1 degree \(\leq d\) polynomial with arithmetic modulo prime \(p\) contains \(d + 1\) pts.
Another Construction: Interpolation!

For a quadratic, \( a_2 x^2 + a_1 x + a_0 \) hits \((1,3); (2,4); (3,0)\).
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For a quadratic, \( a_2 x^2 + a_1 x + a_0 \) hits \((1, 3); (2, 4); (3, 0)\).
Find \( \Delta_1(x) \) polynomial contains \((1, 1); (2, 0); (3, 0)\).
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Try \((x - 2)(x - 3) \mod 5\).
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$\Delta_1(x) = (x - 2)(x - 3)(3) \pmod{5}$
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\[ \Delta_1(x) = (x-2)(x-3)(3) \mod 5 \] contains \((1,1);(2,0);(3,0)\).

\[ \Delta_2(x) = (x-1)(x-3)(4) \mod 5 \] contains \((1,0);(2,1);(3,0)\).
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Find $\Delta_1(x)$ polynomial contains $(1,1); (2,0); (3,0)$.

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$\Delta_2(x) = (x - 1)(x - 3)(4) \pmod{5}$ contains $(1,0); (2,1); (3,0)$.
$\Delta_3(x) = (x - 1)(x - 2)(3) \pmod{5}$ contains $(1,0); (2,0); (3,1)$.
Another Construction: Interpolation!

For a quadratic, \( a_2x^2 + a_1x + a_0 \) hits \((1, 3); (2, 4); (3, 0)\).

Find \( \Delta_1(x) \) polynomial contains \((1, 1); (2, 0); (3, 0)\).

Try \((x - 2)(x - 3) \mod 5\).

Value is 0 at 2 and 3. Value is 2 at 1. Not 1! Doh!!

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...after a lot of calculations...
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So “Divide by 2” or multiply by 3.

\[
\Delta_1(x) = (x - 2)(x - 3)(3) \pmod{5} \text{ contains } (1,1);(2,0);(3,0).
\]

\[
\Delta_2(x) = (x - 1)(x - 3)(4) \pmod{5} \text{ contains } (1,0);(2,1);(3,0).
\]

\[
\Delta_3(x) = (x - 1)(x - 2)(3) \pmod{5} \text{ contains } (1,0);(2,0);(3,1).
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...after a lot of calculations... \(P(x) = 2x^2 + 1x + 4 \mod{5}\).
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The same as before!
Interpolation: in general.

Given points: \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).
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\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.
\]
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And..

\[
P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_k \Delta_k(x).
\]

hits points \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).
Interpolation: in general.

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Construction proves the existence of a degree \(d\) polynomial!
Interpolation: in pictures.

Points: (1, 3.2), (2, 1.3), (3, 1.8).
Interpolation: in pictures.

Points: \((1,3.2), (2,1.3), (3,1.8)\).

\[ P(x) = 3.2 \Delta_1(x) + 1.3 \Delta_2(x) + 1.8 \Delta_3(x) \]
Interpolation: in pictures.

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Scale each $\Delta_i(x)$ function and add to contain points.

$$P(x) = 3.2 \Delta_1(x) + 1.3 \Delta_2(x) + 1.8 \Delta_3(x)$$
Interpolation: in pictures.

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P(x) = 3.2 \Delta_1(x) + 1.3 \Delta_2(x) + 1.8 \Delta_3(x)
\]
Interpolation: in pictures.

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Scale each $\Delta_i$ function and add to contain points.

$$P(x) = 3.2 \Delta_1(x) + 1.3\Delta_2(x) + 1.8\Delta_3(x)$$
Interpolation and Existence

Interpolation takes $d + 1$ points and produces a degree $d$ polynomial that contains the points.
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Interpolation takes $d+1$ points and produces a degree $d$ polynomial that contains the points.

**Construction proves the existence of a degree $d$ polynomial that contains points!**
Interpolation takes \(d + 1\) points and produces a degree \(d\) polynomial that contains the points.

Construction proves the existence of a degree \(d\) polynomial that contains points!

Is it the only degree \(d\) polynomial that contains the points?
Uniqueness.

**Uniqueness Fact.** At most one degree $d$ polynomial hits $d + 1$ points.
Uniqueness.

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**Proof:**
Uniqueness.

**Uniqueness Fact.** At most one degree $d$ polynomial hits $d+1$ points.

**Proof:**

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.
Uniqueness Fact. At most one degree $d$ polynomial hits $d + 1$ points.

Proof:

Roots fact: Any degree $d$ polynomial has at most $d$ roots.
Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.
Uniqueness Fact. At most one degree $d$ polynomial hits $d + 1$ points.

Proof:

Roots fact: Any degree $d$ polynomial has at most $d$ roots. Assume two different polynomials $Q(x)$ and $P(x)$ hit the points. $R(x) = Q(x) - P(x)$ has $d + 1$ roots and is degree $d$. 
**Uniqueness Fact.** At most one degree $d$ polynomial hits $d + 1$ points.

**Proof:**

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.

Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d + 1$ roots and is degree $d$.

Contradiction.
Uniqueness Fact. At most one degree $d$ polynomial hits $d + 1$ points.

Proof:

Roots fact: Any degree $d$ polynomial has at most $d$ roots.

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Contradiction. □
Uniqueness. 

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**Proof:**

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.

Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d + 1$ roots and is degree $d$.

Contradiction.

Must prove **Roots fact.**
Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.
Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c|cc}
& 4 & x \\
\hline
x - 3 & 4x^2 & - 3x \\
& - 4x^2 & + 12x \\
\hline
& 4 & x \\
& - 4 & + 2 \\
\hline
& 4 & 2
\end{array}
\]

$4x^2 - 3x + 2 \equiv (x - 3) (4x + 4) + 4 \pmod{5}$

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x - a)Q(x) + r$.
Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{r}
\multicolumn{2}{c}{4x} \\
\hline
x - 3 & 4x^2 - 3x + 2 \\
\quad & -(4x^2 - 2x) \\
\hline
\quad & 4x + 2 \\
\quad & -(4x - 2) \\
\hline
\quad & 4 \\
\end{array}
\]

$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}$.

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

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Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{r}
4 & x & + & 4 \\
\hline
x - 3 & ) & 4x^2 & - & 3x & + & 2 \\
& & - & (4x^2 & - & 2x) \\
& & & - & 4x & + & 2 \\
\end{array}
\]

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x - a)Q(x) + r$.
Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c}
\begin{array}{c}
4x + 4 \\
\hline \\
x - 3 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
4x^2 - 3x + 2 \\
- (4x^2 - 2x) \\
\hline \\
4x + 2 \\
- (4x - 2) \\
\end{array}
\end{array}
\]

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4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}
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Polynomial Division.
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\[
\begin{array}{c}
4x + 4 \\
\hline
x - 3 \mid 4x^2 - 3x + 2 \\
- (4x^2 - 2x) \\
\hline
4x + 2 \\
- (4x - 2) \\
\hline
4
\end{array}
\]

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$.

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\[P(x) = (x - a)Q(x) + r\]
Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

\[
\begin{array}{c}
4x + 4 \quad r \quad 4 \\
\hline
x - 3 \quad | \quad 4x^2 - 3x + 2 \\
- \quad (4x^2 - 2x) \\
\hline
4x + 2 \\
- \quad (4x - 2) \\
\hline
4
\end{array}
\]

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$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}$
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\[
\begin{array}{rll}
4x + 4 & r & 4 \\
\hline
x - 3 & | & 4x^2 - 3x + 2 \\
        & - & (4x^2 - 2x) \\
        & --- & \\
        & 4x + 2 \\
        & - & (4x - 2) \\
        & --- & \\
        & 4 \\
\end{array}
\]

$4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}$
Polynomial Division.
Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

$$
\begin{array}{c}
\begin{array}{cccc}
4 & x & + & 4 & r & 4 \\
\hline
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
\hline
x & - & 3 & ) & 4x^2 & - & 3x & + & 2 \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
\hline
- & (4x^2 & - & 2x) \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
\hline
& 4x & + & 2 \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
\hline
- & (4x & - & 2) \\
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{cccc}
\hline
& 4 \\
\end{array}
\end{array}
\end{array}
$$

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\[
\begin{array}{c}
4x + 4 & r & 4 \\
\hline
x - 3 & 4x^2 - 3x + 2 \\
& - (4x^2 - 2x) \\
& \hline
& 4x + 2 \\
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& \hline
& 4
\end{array}
\]

\[4x^2 - 3x + 2 \equiv (x - 3)(4x + 4) + 4 \pmod{5}\]

In general, divide $P(x)$ by $(x - a)$ gives $Q(x)$ and remainder $r$. That is, $P(x) = (x - a)Q(x) + r$
Only $d$ roots.

**Lemma 1:** $P(x)$ has root $a$ iff $P(x)/(x - a)$ has remainder 0: $P(x) = (x - a)Q(x)$. 

Proof: 

$P(x) = (x - a)Q(x) + r$. 

Plugin $a$: $P(a) = r$. It is a root if and only if $r = 0$.

**Lemma 2:** $P(x)$ has $d$ roots; $r_1, \ldots, r_d$ then $P(x) = c(x - r_1)(x - r_2)\cdots(x - r_d)$.

Proof Sketch: 

By induction. 

Induction Step: $P(x) = (x - r_1)Q(x)$ by Lemma 1. 

$P(x) = 0$ if and only if $(x - r_1)$ is 0 or $Q(x) = 0$. 

Root either at $r_1$ or root of $Q(x)$. 

$Q(x)$ has smaller degree and $r_2, \ldots, r_d$ are roots. 

Use the induction hypothesis. 

$d + 1$ roots implies degree is at least $d + 1$. 

Roots fact: Any degree $d$ polynomial has at most $d$ roots.
Lemma 1: $P(x)$ has root $a$ iff $P(x)/(x - a)$ has remainder 0: $P(x) = (x - a)Q(x)$.

Proof: $P(x) = (x - a)Q(x) + r$. Plugin $a$: $P(a) = r$. 

Only $d$ roots.
Only \( d \) roots.

**Lemma 1:** \( P(x) \) has root \( a \) iff \( P(x)/(x - a) \) has remainder 0: 
\[
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\]

**Proof:** \( P(x) = (x - a)Q(x) + r. \)
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**Proof:** $P(x) = (x-a)Q(x) + r$. Plugin $a$: $P(a) = r$. It is a root if and only if $r = 0$.

**Lemma 2:** $P(x)$ has $d$ roots; $r_1, \ldots, r_d$ then

\[ P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d). \]
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$ab = 0 \implies a = 0$ or $b = 0$ in field.
Only \( d \) roots.

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\[ P(x) = (x - a)Q(x). \]

**Proof:** \( P(x) = (x - a)Q(x) + r. \)
Plugin \( a \): \( P(a) = r. \) It is a root if and only if \( r = 0. \)

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\[ P(x) = c(x - r_1)(x - r_2)\cdots(x - r_d). \]

**Proof Sketch:** By induction.
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\( P(x) = 0 \) if and only if \( (x - r_1) \) is 0 or \( Q(x) = 0. \)
\[ ab = 0 \implies a = 0 \text{ or } b = 0 \text{ in field.} \]
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**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.
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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.