Infinity and Uncountability.

- Countable
- Countably infinite.
- Enumeration

How big is the set of reals or the set of integers?

Infinite!

Is one bigger or smaller?

Same size?

Same number?

Make a function \( f : \text{Circles} \rightarrow \text{Squares} \).

\( f(\text{red circle}) = \text{red square} \)

\( f(\text{blue circle}) = \text{blue square} \)

\( f(\text{circle with black border}) = \text{square with black border} \)

One to one:
Each circle mapped to different square.

One to One: For all \( x, y \in D \), \( x \neq y \Rightarrow f(x) \neq f(y) \).

Onto:
Each square mapped to from some circle.

Onto: For all \( s \in R \), \( \exists c \in D \), \( s = f(c) \).

Isomorphism principle:
If there is \( f : D \rightarrow R \) that is one to one and onto, then, \( |D| = |R| \).

Combinatorial Proofs.

The number of subsets of a set \( \{a_1, \ldots, a_n\} \)?

Equal to the number of binary \( n \)-bit strings.

\( f : \text{Subsets} \rightarrow \text{Strings} \).

\( f(x) = (g(x, a_1), g(x, a_2), \ldots, g(x, a_n)) \)

\( g(x, a) = \begin{cases} 1 & a \in x \\ 0 & \text{otherwise} \end{cases} \)

Example:
\( S = \{1,2,3,4,5\}, x = \{1,3,4\} \).

\( f(x) = (1,0,1,1,0) \).

\( |P(S)| = |\{0,1\}^5| = 2^5 \).

Countable.

How to count?

0, 1, 2, 3, 

The Counting numbers.

The natural numbers! \( N \)

Definition: \( S \) is countable if there is a bijection between \( S \) and some subset of \( N \).

If the subset of \( N \) is finite, \( S \) has finite cardinality.

If the subset of \( N \) is infinite, \( S \) is countably infinite.
Where's 0?
Which is bigger?
The positive integers, $Z^+$, or the natural numbers, $N$.
Natural numbers. 0, 1, 2, 3, …
Positive integers. 1, 2, 3, …
Where's 0?
More natural numbers!
Consider $f : Z^+ \to N$ where $f(z) = z - 1$.
For any two $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$.
One to one!
For any natural number $n$,
for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.
Onto!
Bijection!
|$Z^+$| = |$N$|.
But... but where's zero? "It comes from 1."

A bijection is a bijection.
Notice that there is a bijection between $N$ and $Z^+$ as well.
\[ f(n) = n + 1 \textrm{.} \quad 0 \to 1, 1 \to 2, \ldots \]
Bijection from $A$ to $B$ \iff a bijection from $B$ to $A$.
Inverse function!
Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

Listings..
\[ f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases} \]
Another View:
\[
\begin{array}{c|c}
 n & f(n) \\
\hline
 0 & 0 \\
 1 & 1 \\
 2 & 1 \\
 3 & -2 \\
 4 & 2 \\
\end{array}
\]
Notice that: A listing "is" a bijection with a subset of natural numbers.
Function = "Position in list."
If finite: bijection with $\{0, \ldots, |S| - 1\}$
If infinite: bijection with $N$.

More large sets.
\[ E \subseteq \text{Even natural numbers?} \]
\[ f : N \to E. \]
\[ f(n) \to 2n. \]
Onto: \forall e \in E, f(e/2) = e. e/2 is natural since $e$ is even.
One-to-one: \forall x, y \in N, x \neq y \implies 2x \neq 2y. \implies f(x) \neq f(y)

Enumerability $\equiv$ countability.
Enumerating (listing) a set implies that it is countable.
"Output element of $S$",
"Output next element of $S$"
Any element $x$ of $S$ has specific, finite position in list.
\[ Z = \{0, 1, 2, \ldots\} \]
\[ Z = \{0, 1, 2, \ldots\} \text{ and then } \{-1, -2, \ldots\} \]
When do you get to $-1$? at infinity?
Need to be careful.

Integers and naturals have same size!
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Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset $T$ of a countable set $S$ is countable.
Enumerate $T$ as follows:
Get next element, $x$, of $S$,
output only if $x \in T$.

Implications:
$\mathbb{Z}^+$ is countable.
It is infinite.
There is a bijection with the natural numbers.
So it is countably infinite.
All countably infinite sets have the same cardinality.

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
E.g.: (1,2), (100,30), etc.
For finite sets $S_1$ and $S_2$,
then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.
So, is $N \times N$ countably infinite squared ???

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Pairs of natural numbers.

Enumerate in list:
(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),......
0
1
2
3
0 1 2 3 4
· · · · ... (a + b + 1)(a + b)/2 elements of list!
(i.e., "triangle").
Countably infinite.
Same size as the natural numbers!!

Fractions?

Can you enumerate the rational numbers in order?
0,..., 1/2...
Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
A thing about fractions:
any two fractions has another fraction between it.
Does this mean we can’t even get to “next” fraction?
Can’t list in “order”?

Rationals?

Positive rational number.
Lowest terms: $a/b$
$a, b \in N$
with $\gcd(a,b) = 1$.
Infinite subset of $N \times N$.
Countably infinite!
All rational numbers?
Negative rationals are countable. (Same size as positive rationals.)
Put all rational numbers in a list.
First negative, then nonegative ??? No!
Repeatedly and alternatively take one from each list.
Interleave Streams in 61A
The rationals are countably infinite!
### Real numbers.

Is the set of real numbers the “same size” as integers?

### The reals.

Are the set of reals countable?

- Each real has a decimal representation.
- Some real number

### Diagonalization.

If countable, there a listing, $L$ contains all reals. For example:

- 0: .500000000...
- 1: .785398162...
- 2: .367879441...
- 3: .632120558...
- 4: .345212312...

Construct “diagonal” number: $0.7777777...$

Diagonal Number: Digit $i$ is 7 if number $i$’s, $i$th digit is not 7 and 6 otherwise.

- Diagonal number for a list differs from every number in list!
- Diagonal number not in list.
- Diagonal number is real.
- Contradiction!

- Subset $[0,1]$ is not countable!!

### All reals?

- Subset $[0,1]$ is not countable!!
- What about all reals?
- No.

Any subset of a countable set is countable.

If reals are countable then so must $[0,1]$.

### Diagonalization: Summary

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.
4. Show that $t$ is different from all elements in the list $\Rightarrow t$ is not in the list.
5. Show that $t$ is in $S$.
6. Contradiction.

### Another diagonalization.

The set of all subsets of $N$.

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:

- If $i$th set in $L$ does not contain $i$, $i \in D$.
- Otherwise $i \notin D$.

$D$ is different from $i$th set in $L$ for every $i$.

$D$ is a subset of $N$.

$L$ does not contain all subsets of $N$.

Contradiction.

**Theorem:** The set of all subsets of $N$ is not countable. (The “set of all subsets of $N$” is the powerset of $N$.)
Cardinalities of uncountable sets?

Cardinality of \([0,1]\) smaller than all the reals?

\[ f : R^+ \to [0,1]. \]

\[ f(x) = \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq 1/2 \\
  \frac{1}{x} & x > 1/2 
\end{cases} \]

One to one, \(x \neq y\)

- If both in \([0,1/2]\), a shift \(\implies f(x) \neq f(y)\).
- If neither in \([0,1/2]\) a division \(\implies f(x) \neq f(y)\).
- If one is in \([0,1/2]\) and one isn’t, different ranges \(\implies f(x) \neq f(y)\).

Bijection!

\([0,1]\) is same cardinality as nonnegative reals!

Summary.

- Bijections to equate cardinality of infinite sets
- Countable (infinite) sets
- Uncountable sets
- Diagonalization