Bayes’ Rule, Independence, Mutual Independence

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Bayes and Biased Coin

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Conditional Probability: Review

Recall:
- \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \)
- Hence, \( P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A) \).
- \( A \) and \( B \) are positively correlated if \( P(A \mid B) > P(A) \), i.e., if \( P(A \cap B) > P(A)P(B) \).
- \( A \) and \( B \) are negatively correlated if \( P(A \mid B) < P(A) \), i.e., if \( P(A \cap B) < P(A)P(B) \).
- Note: \( B \subset A \Rightarrow A \) and \( B \) are positively correlated. \( P(A \cap B) = P(B) > P(A) \).
- Note: \( A \cap B = \emptyset \Rightarrow A \) and \( B \) are negatively correlated. \( P(A \cap B) = 0 < P(A) \).

Bayes and Biased Coin

Bayes: General Case

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Bayes Rule

Another picture:

Pick a point uniformly at random in the unit square. Then
\[
\begin{align*}
P[A] &= 0.5; \quad P[\overline{A}] = 0.5 \\
P[B] &= 0.5 \times 0.5 + 0.5 \times 0.6 = P[A]P[B \mid A] + P[\overline{A}]P[B \mid \overline{A}] \\
&= 0.5 \times 0.5 + 0.5 \times 0.6 \\
&= 0.5 \times 0.5 + 0.5 \times 0.6 = P[A \mid P[B \mid A] + P[\overline{A}]P[B \mid \overline{A}] \\
&= 0.46 \text{ fraction of } B \text{ that is inside } A
\end{align*}
\]
What is the posterior probability of having a fever given that 'Flu' is the MAP cause of high fever?

We found:

\[
\begin{align*}
\Pr[\text{Flu} | \text{High Fever}] &= 0.58, \\
\Pr[\text{Ebola} | \text{High Fever}] &= 5 \times 10^{-8}, \\
\Pr[\text{Other} | \text{High Fever}] &= 0.42.
\end{align*}
\]

Note that even though \( \Pr[\text{Fever} | \text{Ebola}] = 1 \), one has

\[
\Pr[\text{Ebola} | \text{Fever}] \approx 0.
\]

This example shows the importance of the prior probabilities.

**Independence**

**Definition:** Two events \( A \) and \( B \) are independent if

\[
\Pr[A \cap B] = \Pr[A] \Pr[B].
\]

**Examples:**

- When rolling two dice, \( A = \text{sum is 7} \) and \( B = \text{red die is 1} \) are independent;
- When rolling two dice, \( A = \text{sum is 3} \) and \( B = \text{red die is 1} \) are not independent;
- When flipping coins, \( A = \text{coin 1 yields heads} \) and \( B = \text{coin 2 yields tails} \) are independent;
- When throwing 3 balls into 3 bins, \( A = \text{bin 1 is empty} \) and \( B = \text{bin 2 is empty} \) are not independent;

**Independence and conditional probability**

**Fact:** Two events \( A \) and \( B \) are independent if and only if

\[
\Pr[A | B] = \Pr[A].
\]

Indeed:

\[
\begin{align*}
\Pr[A | B] &= \frac{\Pr[A \cap B]}{\Pr[B]} \\
&= \frac{\Pr[A] \Pr[B]}{\Pr[B]} \\
&= \Pr[A],
\end{align*}
\]

Thus, \( A \) and \( B \) are independent if and only if

\[
\Pr[A | B] = \Pr[A].
\]

Consider the example below:

\[
(A_3, B) \text{ are independent: } \Pr[A_3 | B] = 0.5 = \Pr[A_3].
\]

\[
(A_3, B) \text{ are independent: } \Pr[A_3 | B] = 0.5 = \Pr[A_3].
\]

\[
(A_3, B) \text{ are not independent: } \Pr[A_3 | B] = \frac{0.25}{0.25} = 0.2 \neq \Pr[A_3] = 0.25.
\]
Example 2

Flip a fair coin 5 times. Let $A_n = \text{`coin n is H'}, \text{for } n = 1, \ldots, 5.$ Then, $A_m, A_n$ are independent for all $m \neq n.$

Also, $A_1$ and $A_3 \cap A_5$ are independent.

Indeed,

$$\Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = \Pr[A_1] \Pr[A_3 \cap A_5].$$

Similarly,

$$A_1 \cap A_2 \text{ and } A_3 \cap A_4 \cap A_5 \text{ are independent.}$$

This leads to a definition ....

Mutual Independence

**Definition** Mutual Independence

(a) The events $A_1, \ldots, A_n$ are mutually independent if

$$\Pr[\bigcap_{k \in K} A_k] = \prod_{k \in K} \Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}. $$

(b) More generally, the events $\{A_j \mid j \in J\}$ are mutually independent if

$$\Pr[\bigcap_{k \in K} A_k] = \prod_{k \in K} \Pr[A_k], \text{ for all finite } K \subseteq J.$$

Thus, $\Pr[A_1 \cap A_2] = \Pr[A_1] \Pr[A_2], \Pr[A_1 \cap A_3 \cap A_4 \cap A_5] = \Pr[A_1] \Pr[A_3] \Pr[A_4] \Pr[A_5] \cdots \Pr[A_n].$ Example: Flip a fair coin forever. Let $A_n = \text{`coin n is H'}.$ Then the events $A_n$ are mutually independent.

Summary.

- **Bayes’ Rule** $\Pr[A_i | B] = \frac{\Pr[A_i] \Pr[B]}{\Pr[B]}.$
- Mutual Independence: Events defined by disjoint collections of mutually independent events are mutually independent.