CS70: Lecture 3. Induction!

1. The natural numbers.
2. Seven year old Gauss.
3. ...and Induction.
4. Simple Proof.
5. Two coloring map
1. The natural numbers.
2. Seven year old Gauss.
3. ...and Induction.
4. Simple Proof.
5. Two coloring map

(mostly) Next time:
2. Tiling Cory Hall courtyard.
3. Horses with one color...
The naturals.
The naturals.
The naturals.
The naturals.

0, 1,
The naturals.

0, 1, 2,
The naturals.

0, 1, 2, 3,
The naturals.

0, 1, 2, 3, ...

0, 1, 2, 3, ...

The naturals.

0, 1, 2, 3, ..., n,
The naturals.

0, 1, 2, 3, ..., n, n+1,
The naturals.

0, 1, 2, 3, 
..., n, n+1, n+2, n+3,
The naturals.

0, 1, 2, 3, ...
... n, n+1, n+2, n+3, ...

A Story about a 7-year old Gauss.
A Story about a 7-year old Gauss.

Teacher: Hello class.
A Story about a 7-year old Gauss.

Teacher: Hello class.
Teacher:

\[\text{Teacher: Please add the numbers from 1 to 100. Gauss: It's 5050! (that is, } \frac{100 \times (100+1)}{2} \text{)}\]
A Story about a 7-year old Gauss.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s 5050!
A Story about a 7-year old Gauss.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It's 5050! (that is, \( \frac{100 \times 101}{2} \))
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\)
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N}) \left( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right) \) Proof?

Idea: assume predicate for \( n = k \).

\[
\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.
\]

Is predicate true for \( n = k + 1 \)?

\[
\sum_{i=1}^{k+1} i = \left( \sum_{i=1}^{k} i \right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.
\]

How about \( k + 2 \).

Same argument starting at \( k + 1 \) works!

Induction Step.

Is this a proof?

It shows that we can always move to the next step.

Need to start somewhere.

\[
\sum_{i=1}^{1} i = 1 \quad \text{Proof by Induction.}
\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume \textit{predicate} for \(n = k\).
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k+1\)?
\[
\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.
\]

How about \(k+2\).
\[
\sum_{i=1}^{k+2} i = \left(\sum_{i=1}^{k+1} i\right) + (k+2) = \frac{(k+1)(k+2)}{2} + (k+2) = \frac{(k+1)(k+2) + 2(k+2)}{2} = \frac{(k+2)(k+3)}{2}.
\]

Same argument starting at \(k+1\) works!

Induction Step.

Is this a proof?

It shows that we can always move to the next step.

Need to start somewhere.

\[
\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} = \frac{2}{2}.
\]

Base Case.

Statement is true for \(n = 0\) plus inductive step \(\Rightarrow\) true for \(n = 1\) plus inductive step \(\Rightarrow\) true for \(n = 2\)...

true for \(n = k\) \(\Rightarrow\) true for \(n = k+1\)...

Predicate True for all natural numbers!

Proof by Induction.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\(\sum_{i=1}^{k+1} i\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1)\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
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Gauss and Induction

Child Gauss: $(\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate for $n = k$. $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Is predicate true for $n = k + 1$?

$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

How about $k + 2$. 

Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!
Gauss and Induction

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**Induction Step.**
Gauss and Induction

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Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}\].

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof?
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \) Proof?

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How about \( k + 2 \). Same argument starting at \( k + 1 \) works!

Induction Step.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere.
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! **Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[
\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\) plus inductive step
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume **predicate** for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

\[\text{plus inductive step } \implies \text{ true for } n = 1\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

- plus inductive step \(\implies\) true for \(n = 1\)
- plus inductive step
Gauss and Induction

Child Gauss: \( (\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \) Proof?

Idea: assume predicate for \( n = k \). \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

Is predicate true for \( n = k + 1 \)?

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \( k + 2 \). Same argument starting at \( k + 1 \) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \( \sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2} \) **Base Case.**

Statement is true for \( n = 0 \)

   plus inductive step \( \implies \) true for \( n = 1 \)

   plus inductive step \( \implies \) true for \( n = 2 \)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

  plus inductive step \(\Rightarrow\) true for \(n = 1\)

  plus inductive step \(\Rightarrow\) true for \(n = 2\)

  \(...\)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)

plus inductive step \(\implies\) true for \(n = 1\)

plus inductive step \(\implies\) true for \(n = 2\)

\[
\vdots
\]

true for \(n = k\)
**Gauss and Induction**

Child Gauss: \( (\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \) Proof?

Idea: assume predicate for \( n = k \). \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
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How about \( k + 2 \). Same argument starting at \( k + 1 \) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \( \sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2} \) **Base Case.**

Statement is true for \( n = 0 \)

- plus inductive step \( \implies \) true for \( n = 1 \)
- plus inductive step \( \implies \) true for \( n = 2 \)

\[ \ldots \]

- true for \( n = k \) \( \implies \) true for \( n = k + 1 \)
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume predicate for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.\]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

**Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) Base Case.

Statement is true for \(n = 0\)

plus inductive step \(\implies\) true for \(n = 1\)

plus inductive step \(\implies\) true for \(n = 2\)

\[\vdots\]

true for \(n = k \implies\) true for \(n = k + 1\)

\[\vdots\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

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Is predicate true for \(n = k + 1\)?
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\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}.
\]

How about \(k + 2\). Same argument starting at \(k + 1\) works! **Induction Step.**

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) **Base Case.**

Statement is true for \(n = 0\)
\[
\text{plus inductive step } \implies \text{ true for } n = 1
\]
\[
\text{plus inductive step } \implies \text{ true for } n = 2
\]
\[
\ldots
\]
\[
\text{true for } n = k \implies \text{ true for } n = k + 1
\]
\[
\ldots
\]
Gauss and Induction

Child Gauss: \((\forall n \in \mathbb{N}) (\sum_{i=1}^{n} i = \frac{n(n+1)}{2})\) Proof?

Idea: assume \textbf{predicate} for \(n = k\). \(\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\).

Is predicate true for \(n = k + 1\)?

\[ \sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k + 1) = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}. \]

How about \(k + 2\). Same argument starting at \(k + 1\) works!

\textbf{Induction Step.}

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \(\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}\) \textbf{Base Case.}

Statement is true for \(n = 0\)

\[ \begin{align*}
\text{plus inductive step} & \implies \text{true for } n = 1 \\
\text{plus inductive step} & \implies \text{true for } n = 2 \\
\vdots \\
\text{true for } n = k & \implies \text{true for } n = k + 1 \\
\vdots
\end{align*} \]

Predicate \textbf{True} for all natural numbers!

\textbf{Proof by Induction.}
The canonical way of proving statements of the form

\[(\forall k \in N)(P(k))\]
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$. 
Induction

The canonical way of proving statements of the form

\[(\forall k \in N)(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \(n \in N\), \(n^3 - n\) is divisible by 3.
Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.
Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \(n \in \mathbb{N}\), \(n^3 - n\) is divisible by 3.
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The basic form
Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbb{N})(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.

Induction

The canonical way of proving statements of the form

\[(\forall k \in \mathbb{N})(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \(n \in \mathbb{N}\), \(n^3 - n\) is divisible by 3.
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The basic form

- Prove \(P(0)\). “Base Case”.
- \(P(k) \implies P(k+1)\)
The canonical way of proving statements of the form

\[(\forall k \in N)(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
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- The sum of the first \(n\) odd integers is a perfect square.

The basic form

- Prove \(P(0)\). “Base Case”.
- \(P(k) \implies P(k+1)\)
  - Assume \(P(k)\), “Induction Hypothesis”
Induction

The canonical way of proving statements of the form

\[(\forall k \in N)(P(k))\]

- For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\).
- For all \(n \in N\), \(n^3 - n\) is divisible by 3.
- The sum of the first \(n\) odd integers is a perfect square.

The basic form

- Prove \(P(0)\). “Base Case”.
- \(P(k) \implies P(k + 1)\)
  - Assume \(P(k)\), “Induction Hypothesis”
  - Prove \(P(k + 1)\). “Induction Step.”
Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first $n$ odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
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$P(n)$ true for all natural numbers $n$!!!
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Notes visualization

An visualization: an infinite sequence of dominos.

Prove they all fall down;
An visualization: an infinite sequence of dominos.

Prove they all fall down;

- $P(0) = \text{“First domino falls”}$
An visualization: an infinite sequence of dominos.

Prove they all fall down;

- \( P(0) \) = “First domino falls”
- \( (\forall k) P(k) \implies P(k+1) \):
An visualization: an infinite sequence of dominos.

Prove they all fall down;

- \( P(0) = \) “First domino falls”
- \( (\forall k) \ P(k) \implies P(k+1): \) “\( k \)th domino falls implies that \( k+1 \)st domino falls”
Climb an infinite ladder?
Climb an infinite ladder?

Your favorite example of "forever"...
or the integers...
Climb an infinite ladder?

$P(0)$

$P(n)$

$P(n+1)$

$P(n) \Rightarrow P(n+1)$

$(\forall n \in \mathbb{N}) P(n)$

Your favorite example of "forever"... or the integers...
Climb an infinite ladder?

\[
P(0) \\
P(k) \implies P(k+1)
\]
Climb an infinite ladder?

\[
P(0) \\
P(1) \\
P(2)
\]

\[P(k) \implies P(k + 1)\]
Climb an infinite ladder?

\[ P(0) \]
\[ P(1) \]
\[ P(2) \]
\[ P(3) \]

\[ P(n) \Rightarrow P(n+1) \quad (\forall n \in \mathbb{N}) \]

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\[ P(n) \implies P(n+1) \]

Your favorite example of “forever”... or the integers...
Climb an infinite ladder?

\[
P(n) \quad P(n+1) \quad P \cdot P \cdot P \cdot P(n)
\]

Your favorite example of "forever"... or the integers...

\[
P(0) \quad P(k) \quad P(k + 1)
\]

\[
\forall n \in \mathbb{N} \quad P(n)
\]
Climb an infinite ladder?

\[ P(0) \]
\[ P(1) \]
\[ P(2) \]
\[ P(3) \]
\[ P(n) \]
\[ P(n+1) \]

\[ P(0) \Rightarrow P(k) \Rightarrow P(k+1) \]

\( \forall n \in \mathbb{N} \)

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Climb an infinite ladder?

\[
P(0) \quad P(1) \quad P(2) \quad P(3) \quad P(n) \quad P(n+1) \quad P(\ldots)
\]

\[
P(k) \quad P(k+1)
\]

\[
\forall n \in \mathbb{N} \quad P(n)
\]

Your favorite example of "forever"... or the integers...
Climb an infinite ladder?

\[
P(0) \implies P(1) \implies P(2) \implies \ldots
\]

\[
P(n) \implies P(n+1) \implies P(n+2) \implies \ldots
\]

\[
(\forall n \in \mathbb{N}) P(n)
\]
Climb an infinite ladder?

\[ P(0), P(1), P(2), P(3), \ldots \]

\[ P(n) \implies P(n+1) \]

\[ \forall n \in \mathbb{N} \ P(n) \]

Your favorite example of “forever”...
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Simple induction proof.

**Theorem:** For all natural numbers $n$, $1 + 2 \cdots n = \frac{n(n+1)}{2}$
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Base Case: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Hypothesis: $1 + \cdots + n = \frac{n(n+1)}{2}$
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\[
1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)
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$$1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n^2 + n + 2(n+1)}{2}$$
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\[
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\[
= \frac{n^2 + 3n + 2}{2}
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Simple induction proof.

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\[
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\]
\[
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\]
\[
= \frac{n^2 + 3n + 2}{2}
\]
\[
= \frac{(n+1)(n+2)}{2}
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$P(n+1)!$
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\((\forall n \in N) \ (P(n) \implies P(n+1))\).
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$(\forall n \in \mathbb{N}) \ (P(n) \implies P(n+1))$. \qed
Four Color Theorem.

**Theorem:** Any map can be colored so that those regions that share an edge have different colors.
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

Fact: Swapping red and blue gives another valid coloring.
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Two color theorem: proof illustration.

Base Case.
Two color theorem: proof illustration.

Base Case.

1. Add line.
2. Get inherited color for split regions.
3. Switch on one side of new line. (Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \Rightarrow P(k+1)$. 
Two color theorem: proof illustration.

1. Add line.

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Algorithm gives $P(k) \implies P(k + 1)$. 
Summary: principle of induction.

\((P(0)\)
Summary: principle of induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1))))\]
Summary: principle of induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))\]
Summary: principle of induction.

\[
(P(0) \land ((\forall k \in \mathbb{N})(P(k) \implies P(k+1)))) \implies (\forall n \in \mathbb{N})(P(n))
\]

Variations:

\[
(P(0) \land ((\forall n \in \mathbb{N})(P(n) \implies P(n+1)))) \implies (\forall n \in \mathbb{N})(P(n))
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Summary: principle of induction.

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\[(P(0) \land ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))\]
\[(P(1) \land ((\forall n \in N)((n \geq 1) \land P(n)) \implies P(n+1))))\]
Summary: principle of induction.

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Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Summary: principle of induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))\]

Variations:
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Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Base Case: Prove \(P(n_0)\).
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Variations:
\[(P(0) \land ((\forall n \in \mathbb{N})(P(n) \implies P(n + 1)))) \implies (\forall n \in \mathbb{N})(P(n))\]
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Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)

Base Case: Prove \(P(n_0)\).

Ind. Step: Prove.
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\[(P(0) \land \left( (\forall k \in N)(P(k) \implies P(k+1)) \right)) \implies (\forall n \in N)(P(n))\]

Variations:

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Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)

Base Case: Prove \(P(n_0)\).

Ind. Step: Prove. For all values, \(n \geq n_0\),
Summary: principle of induction.

\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]

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Statement to prove: \(P(n)\) for \(n\) starting from \(n_0\)
Base Case: Prove \(P(n_0)\).
Ind. Step: Prove. For all values, \(n \geq n_0\), \(P(n) \implies P(n + 1)\).
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Base Case: Prove \(P(n_0)\).
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Statement is proven!