Stable Marriage Problem


Proved useful in many settings, led eventually to 2012 Nobel Prize in Economics (to Shapley and Roth).

Original Problem Setting:
- Small town with \( n \) men and \( n \) women.
- Each woman has a ranked preference list of men.
- Each man has a ranked preference list of women.

How should they be matched?

What criteria to use?
- Maximize number of first choices.
- Minimize difference between preference ranks.
- Look for stable matchings

Stability.

Consider the couples:
- Alice and Bob
- Mary and John

Bob prefers Mary to Alice.
Mary prefers Bob to John.
Uh...oh! Unstable pairing.

So..

Produce a pairing where there is no running off!

Definition: A **pairing** is disjoint set of \( n \) man-woman pairs.

Example: A pairing \( S = \{ (Bob, Alice); (John, Mary) \} \).

Definition: A rogue couple \( b, g \) for a pairing \( S \):
- \( b \) and \( g \) prefer each other to their partners in \( S \)
Example: Bob and Mary are a rogue couple in \( S \).

A stable pairing??

Given a set of preferences.
Is there a stable pairing?
How does one find it?

Consider a variant of this problem: stable roommates.

\[
\begin{array}{cccc}
A & B & C & D \\
B & C & A & D \\
C & A & B & D \\
D & A & B & C \\
\end{array}
\]

The Stable Marriage Algorithm.

Each Day:
1. Each man **proposes** to his favorite woman on his list.
2. Each woman rejects all but her favorite proposer (whom she puts on a **string**.)
3. Rejected man **crosses** rejecting woman off his list.

Stop when each woman gets exactly one proposal.
Does this terminate?
...produce a pairing?
.....a stable pairing?
Do men or women do “better”?
Example.

<table>
<thead>
<tr>
<th>Men</th>
<th>Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1X</td>
</tr>
<tr>
<td>B</td>
<td>2X</td>
</tr>
<tr>
<td>C</td>
<td>3X</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B</td>
<td>X</td>
<td>A, C</td>
<td>B, C</td>
<td>A, C</td>
</tr>
</tbody>
</table>

Termination.

Every non-terminated day a man crossed an item off the list.

Total size of lists? \( n \) men, \( n \) length list. \( n^2 \)
Terminates in at most \( n^2 + 1 \) steps!

Pairing when done.

**Lemma:** Every man is matched at end.

**Proof:**
If not, a man \( b \) must have been rejected \( n \) times.
Every woman has been proposed to by \( b \), and Improvement lemma
\( \implies \) each woman has a man on a string.
and each man on at most one string.
\( n \) women and \( n \) men. Same number of each.
\( \implies \) \( b \) must be on some woman’s string!

Contradiction.

Pairing is Stable.

**Lemma:** There is no rogue couple for the pairing formed by stable marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*\) \)

\( b' \) ——— \( g^* \)

\( b \) likes \( g^* \) more than \( g \).

\( b \) ——— \( g \)

\( g^* \) likes \( b \) more than \( b' \).

Man \( b \) proposes to \( g^* \) before proposing to \( g \).
So \( g^* \) rejected \( b \) (since he moved on)
By improvement lemma, \( g^* \) likes \( b' \) better than \( b \).

Contradiction!

Good for men? women?

Is the SMA better for men? for women?

**Definition:** A pairing is \( x \)-optimal if \( x \)'s partner
is its best partner in any stable pairing.

**Definition:** A pairing is \( x \)-pessimal if \( x \)'s partner
is its worst partner in any stable pairing.

**Definition:** A pairing is man optimal if it is \( x \)-optimal for all men \( x \).
...and so on for man pessimal, woman optimal, woman pessimal.

Claim: The optimal partner for a man must be first in his preference list.
True? False? False!

Subtlety here: Best partner in any stable pairing.
As well as you can in a globally stable solution!

**Question:** Is there a even man or woman optimal pairing?
SMA is optimal!

For men? For women?

**Theorem: SMA produces a man-optimal pairing.**

**Proof:**

Assume not: there are men who do not get their optimal woman.

Let $t$ be first day any man $b$ gets rejected by his optimal woman $g$ who he is paired with in some stable pairing $S$.

Let $g$ put $b^*$ on a string in place of $b$ on day $t \implies g$ prefers $b^*$ to $b$.

By choice of day $t$, $b^*$ has not yet been rejected by his optimal woman.

Therefore, $b^*$ prefers $g$ to optimal woman, and hence to his partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing. Contradiction.

Recap: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...

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How about for women?

**Theorem: SMA produces woman-pessimal pairing.**

$T$ – pairing produced by SMA.

$S$ – worse stable pairing for woman $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair. $b$ is paired with someone else, say $g^*$.

$g$ likes $b^*$ less than she likes $b$.

$T$ is man optimal, so $b$ likes $g$ more than $g^*$, his partner in $S$.

$(g, b)$ is Rogue couple for $S$

$S$ is not stable.

Contradiction.

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Residency Matching..

The method was used to match residents to hospitals. Hospital optimal... until 1990's... Resident optimal.

Variations: couples!

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Fun stuff from the Fall 2014 offering...

Follow the link.