Stable Marriage Problem

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Proved useful in many settings, led eventually to 2012 Nobel Prize in Economics (to Shapley and Roth).
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Original Problem Setting:
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- Small town with $n$ men and $n$ women.
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Original Problem Setting:

- Small town with \( n \) men and \( n \) women.
- Each woman has a ranked preference list of men.
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Original Problem Setting:

- Small town with $n$ men and $n$ women.
- Each woman has a ranked preference list of men.
- Each man has a ranked preference list of women.
Stable Marriage Problem


Proved useful in many settings, led eventually to 2012 Nobel Prize in Economics (to Shapley and Roth).

Original Problem Setting:

- Small town with $n$ men and $n$ women.
- Each woman has a ranked preference list of men.
- Each man has a ranked preference list of women.

How should they be matched?
What criteria to use?

- Maximize number of first choices.
- Minimize difference between preference ranks.
- Look for stable matchings.
What criteria to use?

- Maximize number of first choices.
What criteria to use?

- Maximize number of first choices.
- Minimize difference between preference ranks.
What criteria to use?

- Maximize number of first choices.
- Minimize difference between preference ranks.
- Look for stable matchings
Stability.

Consider the couples:

- Alice and Bob
- Mary and John
Stability.

Consider the couples:

- Alice and Bob
- Mary and John

Bob prefers Mary to Alice.
Stability.

Consider the couples:
  ▶ Alice and Bob
  ▶ Mary and John

Bob prefers Mary to Alice.
Mary prefers Bob to John.
Stability.

Consider the couples:

- Alice and Bob
- Mary and John

Bob prefers Mary to Alice.
Mary prefers Bob to John.
Uh...oh! Unstable pairing.
So..

Produce a pairing where there is no running off!
So..

Produce a pairing where there is no running off!

**Definition:** A **pairing** is disjoint set of \( n \) man-woman pairs.
So..

Produce a pairing where there is no running off!

**Definition:** A *pairing* is disjoint set of $n$ man-woman pairs.

Example: A pairing $S = \{(Bob, Alice); (John, Mary)\}$. 
Produce a pairing where there is no running off!

**Definition:** A *pairing* is disjoint set of $n$ man-woman pairs.

Example: A pairing $S = \{(Bob, Alice); (John, Mary)\}$.

**Definition:** A *rogue couple* $b, g$ for a pairing $S$: $b$ and $g$ prefer each other to their partners in $S$.
Produce a pairing where there is no running off!

**Definition:** A pairing is disjoint set of $n$ man-woman pairs.

Example: A pairing $S = \{(Bob, Alice); (John, Mary)\}$.

**Definition:** A rogue couple $b, g$ for a pairing $S$: $b$ and $g$ prefer each other to their partners in $S$.

Example: Bob and Mary are a rogue couple in $S$. 
A stable pairing??

Given a set of preferences.
A stable pairing??

Given a set of preferences.
Is there a stable pairing?
How does one find it?
A stable pairing??

Given a set of preferences.

Is there a stable pairing? How does one find it?

Consider a variant of this problem: stable roommates.

\[
\begin{array}{cccc}
A & B & C & D \\
B & C & A & D \\
C & A & B & D \\
D & A & B & C \\
\end{array}
\]

\[\begin{array}{cccc}
A & B \\
B & C \\
C & D \\
D & A \\
\end{array}\]
A stable pairing??

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Is there a stable pairing?
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Consider a variant of this problem: stable roommates.

A | B  C  D
B | C  A  D
C | A  B  D
D | A  B  C

A - B - C - D
A stable pairing??

Given a set of preferences.

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Diagram:

- A -> B
- C -> D
- B -> C
- D -> A
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```
A  B  C  D
B  C  A  D
C  A  B  D
D  A  B  C
```

![Diagram showing paired rooms A and B, and C and D.](attachment:image.png)
A stable pairing??

Given a set of preferences.

Is there a stable pairing?
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C & A & B & D \\
D & A & B & C \\
\end{array}
\]

A \quad B \quad C \quad D 

\[C \quad D\]

A \quad B
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\[
\begin{array}{c}
A \quad \quad B \quad \quad C \quad \quad D \\
B \quad \quad C \quad \quad A \quad \quad D \\
C \quad \quad A \quad \quad B \quad \quad D \\
D \quad \quad A \quad \quad B \quad \quad C \\
\end{array}
\]
The Stable Marriage Algorithm.

Each Day:
1. Each man proposes to his favorite woman on his list.
2. Each woman rejects all but her favorite proposer (whom she puts on a string.)
3. Rejected man crosses rejecting woman off his list.

Stop when each woman gets exactly one proposal.

Does this terminate?

...produce a pairing?

...a stable pairing?

Do men or women do “better”?
The Stable Marriage Algorithm.

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Does this terminate?

...produce a pairing?

....a stable pairing?

Do men or women do “better”? 
Example.

<table>
<thead>
<tr>
<th>Men</th>
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<td>B</td>
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Day 1: A, B
Day 2: A, B, C
Day 3: A, C
Day 4: A
Day 5: B
Example.

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Termination.
Every non-terminated day a man crossed an item off the list.
Every non-terminated day a man **crossed** an item off the list.
Total size of lists?
Termination.

Every non-terminated day a man crossed an item off the list. Total size of lists? $n$ men, $n$ length list.
Termination.

Every non-terminated day a man crossed an item off the list. Total size of lists? $n$ men, $n$ length list. $n^2$
Termination.

Every non-terminated day a man \textbf{crossed} an item off the list. Total size of lists? \(n\) men, \(n\) length list. \(n^2\)
Terminates in at most \(n^2 + 1\) steps!
It gets better every day for women..
It gets better every day for women..

Improvement Lemma:
It gets better every day for women..

**Improvement Lemma:**
If man $b$ proposes to a woman on day $k$,
Improvement Lemma:
If man $b$ proposes to a woman on day $k$, every future day, she has on a string a man $b'$ she likes at least as much as $b$. 
It gets better every day for women..

**Improvement Lemma:**
If man $b$ proposes to a woman on day $k$, every future day, she has on a string a man $b'$ she likes at least as much as $b$. (that is, her options get better)
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**Proof:**
It gets better every day for women..

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**Proof:**
Ind. Hyp.: $P(j) \ (j \geq k)$ — “Woman has as good an option on day $j$ as on day $k$.”
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Base Case: $P(k)$: either she has no one/worse on a string (so puts $b$ or better on a string), or she has someone better already.
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Assume $P(j)$. Let $\hat{b}$ be man on string on day $j \geq k$. So $\hat{b}$ is as good as $b$. 
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On day $j + 1$, man $\hat{b}$ will come back (and possibly others).
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$$\implies P(j + 1).$$
Lemma: Every man is matched at end.
Lemma: Every man is matched at end.

Proof:
Pairing when done.

**Lemma:** Every man is matched at end.

**Proof:**
If not, a man \( b \) must have been rejected \( n \) times.

Every woman has been proposed to by \( b \), and Improvement lemma \( \Rightarrow \) each woman has a man on a string, and each man on at most one string.

Same number of each \( \Rightarrow b \) must be on some woman's string! Contradiction.
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If not, a man $b$ must have been rejected $n$ times.
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$n$ women and $n$ men.
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**Contradiction.**
Pairing is Stable.

**Lemma:** There is no rogue couple for the pairing formed by stable marriage algorithm.
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**Proof:**
Assume there is a rogue couple; \((b, g^*)\)
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\[
\begin{align*}
  b^* & \quad \longrightarrow \quad g^* \\
  b & \quad \longrightarrow \quad g
\end{align*}
\]
Pairing is Stable.

**Lemma:** There is no rogue couple for the pairing formed by stable marriage algorithm.

**Proof:**
Assume there is a rogue couple; \((b, g^*)\)

\[
\begin{array}{c}
  b^* \\
  \text{-----}
\end{array}
\quad
\begin{array}{c}
  g^* \\
  \text{-----}
\end{array}
\quad
\begin{array}{c}
  b \\
  \text{-----}
\end{array}
\quad
\begin{array}{c}
  g
\end{array}
\]

By improvement lemma, \(g^*\) likes \(b^*\) better than \(b\). Contradiction!
Pairing is Stable.

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**Proof:**
Assume there is a rogue couple; $(b, g^*)$

\[
\begin{align*}
  b^* & \longrightarrow g^* & \quad b \text{ likes } g^* \text{ more than } g. \\
  b & \underset{\text{dashed}}{\longrightarrow} g & \quad g^* \text{ likes } b \text{ more than } b^*. 
\end{align*}
\]
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  b & \quad \quad \quad \quad g
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- \(b^*\) likes \(g^*\) more than \(g\).
- \(g^*\) likes \(b\) more than \(b^*\).

Man \(b\) proposes to \(g^*\) before proposing to \(g\).
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 b^* \quad g^* \\
 b \quad g \\
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Man \(b\) proposes to \(g^*\) before proposing to \(g\).
So \(g^*\) rejected \(b\) (since he moved on)
Pairing is Stable.

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\[
\begin{array}{c}
\text{b*} \quad \text{g*} \\
\text{b} \quad \text{g}
\end{array}
\]

- b* likes g* more than g.
- g* likes b more than b*.

Man b proposes to g* before proposing to g.
So g* rejected b (since he moved on)
By improvement lemma, g* likes b* better than b.
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Man \(b\) proposes to \(g^*\) before proposing to \(g\).
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Contradiction!
Good for men? women?

Is the SMA better for men?
Good for men? women?

Is the SMA better for men? for women?

Definition:
A pairing is $x$-optimal if $x$'s partner is its best partner in any stable pairing.

Definition:
A pairing is $x$-pessimal if $x$'s partner is its worst partner in any stable pairing.

Definition:
A pairing is man optimal if it is $x$-optimal for all men $x$.

..and so on for man pessimal, woman optimal, woman pessimal.

Claim: The optimal partner for a man must be first in his preference list.

True? False?

Subtlety here: Best partner in any stable pairing. As well as you can in a globally stable solution!

Question: Is there an even man or woman optimal pairing?
Good for men? women?

Is the SMA better for men? for women?

**Definition:** A **pairing is $x$-optimal** if $x$’s partner is its best partner in any **stable** pairing.
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**Definition:** A *pairing is* \( x \text{-optimal} \) if \( x \)'s partner is its best partner in any *stable* pairing.

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SMA is optimal!

For men?

Theorem: SMA produces a man-optimal pairing.

Proof: Assume not: there are men who do not get their optimal woman. Let $t$ be the first day any man $b$ gets rejected by his optimal woman $g$ who he is paired with in some stable pairing $S$. Let $g$ put $b^*$ on a string in place of $b$ on day $t = \Rightarrow g$ prefers $b^*$ to $b$. By choice of day $t$, $b^*$ has not yet been rejected by his optimal woman. Therefore, $b^*$ prefers $g$ to his optimal woman, and hence to his partner $g^*$ in $S$. Rogue couple for $S$. So $S$ is not a stable pairing. Contradiction.

Recap: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is a rogue couple! Used Well-Ordering principle...
SMA is optimal!

For men? For women?

Theorem:

SMA produces a man-optimal pairing.

Proof:

Assume not: there are men who do not get their optimal woman.

Let $t$ be first day any man $b$ gets rejected by his optimal woman $g$ who he is paired with in some stable pairing $S$.

Let $g$ put $b^*$ on a string in place of $b$ on day $t = \Rightarrow g$ prefers $b^*$ to $b$.

By choice of day $t$, $b^*$ has not yet been rejected by his optimal woman.

Therefore, $b^*$ prefers $g$ to optimal woman, and hence to his partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Contradiction.

Recap:

$S$ - stable. $(b^*, g^*) \in S$.

But $(b^*, g)$ is a rogue couple!

Used Well-Ordering principle...
SMA is optimal!

For men? For women?

**Theorem:** SMA produces a man-optimal pairing.
SMA is optimal!

For men? For women?

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**Proof:**
SMA is optimal!

For men? For women?

**Theorem:** SMA produces a man-optimal pairing.

**Proof:**
Assume not:

Let $t$ be the first day any man $b$ gets rejected by his optimal woman $g$ whom he is paired with in some stable pairing $S$.

Let $g$ put $b^*$ on a string in place of $b$ on day $t$ implies $g$ prefers $b^*$ to $b$.

By choice of day $t$, $b^*$ has not yet been rejected by his optimal woman.

Therefore, $b^*$ prefers $g$ to his optimal woman and hence to his partner $g^*$ in $S$.

Rogue couple for $S$.

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Contradiction.

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Rogue couple for $S$. 
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Let \( g \) put \( b^* \) on a string in place of \( b \) on day \( t \Rightarrow g \) prefers \( b^* \) to \( b \).

By choice of day \( t \), \( b^* \) has not yet been rejected by his optimal woman.

Therefore, \( b^* \) prefers \( g \) to optimal woman, and hence to his partner \( g^* \) in \( S \).

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Therefore, \( b^* \) prefers \( g \) to optimal woman, and hence to his partner \( g^* \) in \( S \).

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Recap:
SMA is optimal!

For men? For women?

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**Rogue couple for $S$.**

So $S$ is not a stable pairing. **Contradiction.**

Recap: $S$ - stable.
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For men? For women?

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Therefore, \( b^* \) prefers \( g \) to optimal woman, and hence to his partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction.

Recap: \( S \) - stable. \((b^*, g^*) \in S\). But \((b^*, g)\) is rogue couple!
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Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Recap: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...
How about for women?

Theorem: SMA produces woman-pessimal pairing.

- T: pairing produced by SMA.
- S: worse stable pairing for woman g.

In T, (g, b) is pair.
In S, (g, b*) is pair. b is paired with someone else, say g*.
g likes b* less than she likes b.
T is man optimal, so b likes g more than g*, his partner in S.

(S, b) is Rogue couple for S. S is not stable.
Contradiction.
How about for women?

**Theorem:** SMA produces woman-pessimal pairing.
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$T$ – pairing produced by SMA.
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How about for women?

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How about for women?

**Theorem:** SMA produces woman-pessimal pairing.

$T$ – pairing produced by SMA.

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$T$ is man optimal, so $b$ likes $g$ more than $g^*$, his partner in $S$. 
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$(g, b)$ is Rogue couple for $S$
How about for women?

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$S$ is not stable.
Theorem: SMA produces woman-pessimal pairing.

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Contradiction.
Residency Matching..
The method was used to match residents to hospitals. Hospital optimal.... ..until 1990’s...Resident optimal. Variations: couples!
Fun stuff from the Fall 2014 offering...

Follow the link.
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