Today.

Types of graphs.

Complete Graphs.
Trees.
Hypercubes.
Complete Graph.

$K_n$ complete graph on $n$ vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n - 1$ edges.
Sum of degrees is $n(n - 1)$.
⇒ Number of edges is $n(n - 1)/2$.
Remember sum of degree is $2|E|$. 
$K_4$ and $K_5$

$K_5$ is not planar.
Cannot be drawn in the plane without an edge crossing!
Prove it! Read Note 5!!
Graph $G = (V, E)$. Binary Tree!

More generally.
Trees: Definitions

Definitions:

- A connected graph without a cycle.
- A connected graph with $|V| - 1$ edges.
- A connected graph where any edge removal disconnects it.
- A connected graph where any edge addition creates a cycle.

Some trees.

- [Diagram of a tree]
- [Diagram of a tree]
- [Diagram of a tree]

no cycle and connected? Yes.
$|V| - 1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check, but yes.
Adding any edge creates cycle. Harder to check, but yes.

Tree or not tree!
Thm:
“$G$ connected and has $|V| - 1$ edges” $\equiv$
“$G$ is connected and has no cycles.”

Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step: Assume for $G$ with up to $k$ vertices. Prove for $k + 1$
Consider some vertex $v$ in $G$. How is it connected to the rest of $G$?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?
Is the rest of $G$ connected?
Theorem:
“G connected and has $|V| - 1$ edges” $\equiv$
“G is connected and has no cycles.”

Lemma: If $v$ is a degree 1 in connected graph $G$, $G - v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,
there is path between $x$ and $y$ in $G$ since connected.
and does not use $v$ (degree 1)
$\implies G - v$ is connected.
Proof of only if.

**Thm:**
“$G$ connected and has $|V| - 1$ edges” $\equiv$
“$G$ is connected and has no cycles.”

**Proof of $\implies$:** By induction on $|V|$.

Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step: Assume for $G$ with up to $k$ vertices. Prove for $k + 1$

**Claim:** There is a degree 1 node.

**Proof:** First, connected $\implies$ every vertex degree $\geq 1$.

- Sum of degrees is $2|V| - 2$
- Average degree $2 - (2/|V|)$
- Not everyone is bigger than average!

By degree 1 removal lemma, $G - v$ is connected.

$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction

$\implies$ no cycle in $G - v$.

And no cycle in $G$ since degree 1 cannot participate in cycle.
Proof of “if part”

Thm:
“G is connected and has no cycles” \(\implies\) “G connected and has \(|V| - 1\) edges”

Proof: Can we use the “degree 1” idea again?
Walk from a vertex using untraversed edges and vertices.
Until get stuck. Why? Finitely-many vertices, no cycle!

Claim: Degree 1 vertex.

Proof of Claim:
Can’t visit more than once since no cycle.
Entered. Didn’t leave. Only one incident edge.
Removing node doesn’t create cycle.
New graph is connected. (from our Degree 1 lemma).
By induction \(G - v\) has \(|V| - 2\) edges.
\(G\) has one more or \(|V| - 1\) edges.
Hypercubes.

Complete graphs, really well connected! Lots of edges.

\[ |V|(|V| - 1)/2 \]

Trees, connected, few edges.

\((|V| - 1)\)

Hypercubes. Well connected. \(|V| \log |V|\) edges!

Also represents bit-strings nicely.

\[ G = (V, E) \]
\[ |V| = \{0,1\}^n, \]
\[ |E| = \{(x,y)|x \text{ and } y \text{ differ in one bit position.}\} \]

2\(^n\) vertices. number of \(n\)-bit strings!

\(n2^{n-1}\) edges.

\(2^n\) vertices each of degree \(n\)

total degree is \(n2^n\) and half as many edges!
A 0-dimensional hypercube is a node labelled with the empty string of bits.

An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$. 
**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$: $|E \cap S \times (V - S)| \geq |S|

Terminology:

$(S, V - S)$ is cut.

$(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**

Base Case: \(n = 1\) \(V = \{0, 1\}\).

- \(S = \{0\}\) has one edge leaving.
- \(S = \emptyset\) has 0.
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:

\[
\begin{align*}
H_0 &= (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.} \\
H &= (V_0 \cup V_1, E_0 \cup E_1 \cup E_x) \\
S &= S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
\end{align*}
\]

**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).

Total cut edges \(\geq |S_0| + |S_1| = |S|\).
**Induction Step. Case 2.**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2. \(|S_0| \geq |V_0|/2\).

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2\).

\[
\implies \geq |S_1| \text{ edges cut in } E_1.
\]

\(|S_0| \geq |V_0|/2 \implies |V_0 - S_0| \leq |V_0|/2\)

\[
\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.
\]

Edges in \(E_x\) connect corresponding nodes.

\[
\implies = |S_0| - |S_1| \text{ edges cut in } E_x.
\]

Total edges cut:

\[
\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\]

\[
|V_0| = |V|/2 \geq |S|.
\]

Also, case 3 where \(|S_1| \geq |V|/2\) is symmetric.
Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)

Central object of study.