1. The goal of this question is to generalize the Chinese remainder theorem to the situation where the two moduli \( m \) and \( n \) do have common factors.

(a) Suppose that \( a, b, n \) are all divisible by \( d \), and let \( a' = a/d \), \( b' = b/d \) and \( n' = n/d \). Show that if \( a \equiv b \mod n \) then \( a' \equiv b' \mod n' \).

(b) Prove that the congruences \( x \equiv a \mod m \) and \( x \equiv b \mod n \) have no solution if \( a \neq b \mod \gcd(m,n) \).

(c) Now show that if \( a \equiv b \mod \gcd(m,n) \) then there is a unique solution for \( x \mod \text{lcm}(m,n) \).

Recall that the \( \text{lcm}(m,n) \) is the least common multiple of \( m \) and \( n \): i.e. the smallest number \( k \) such that \( m \mid k \) and \( n \mid k \). Also \( \gcd(m,n)\text{lcm}(m,n) = mn \).

(d) Now solve the following puzzle from Brahma-Sphuta-Siddhanta (Brahma’s Correct System) by Brahmagupta (born 598 AD):

An old woman goes to market and a horse steps on her basket and crashes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same happened when she picked them out three, four, five, and six at a time, but when she took them seven at a time they came out even. What is the smallest number of eggs she could have had?

2. This time we will generalize the Chinese remainder theorem as proved in class in a different direction. Suppose that the numbers \( n_1, \ldots, n_k \) are pairwise relatively prime; i.e. \( \gcd(n_i, n_j) = 1 \). Prove by induction on \( n \) that the set of congruences \( x \equiv a_i \mod n_i \) have a unique solution \( x \mod n_1 n_2 \cdots n_k \).