Counting and Probability

The topic for the third and final major portion of the course is Discrete Probability. Before we can really explore this subject, we will study the basics of counting. Many of the questions we will be interested in can be cast in the following simple framework:

Balls in Bins: We have a set of k balls. We wish to place them in n bins.

For example, we might have two bins labelled H and T, corresponding to the two outcomes of a coin toss. The balls correspond to the outcomes of successive coin tosses. Or we might consider the situation with 52 bins corresponding to a deck of cards. Here the balls correspond to successive cards in a deal.

The two examples illustrate two different constraints on ball placements. In the coin tossing case, different balls can be placed in the same bin. This is called sampling with replacement. In the cards case, no bin can contain more than one ball. i.e. the same card cannot be dealt twice. This is called sampling without replacement. We will be interested in counting the number of ways of placing k balls in n bins in each of these scenarios.

There are two basic rules of counting that help us do this:

First Rule of Counting: If an object can me made by a succession of choices, where there are n_1 ways of making the first choice, and *for every* way of making the first choice there are n_2 ways of making the second choice, and *for every* way of making the first and second choice there are n_3 ways of making the third choice, and so on up to the n_k-th choice, then the total number of distinct objects that can be made in this way is the product $n_1 \cdot n_2 \cdot n_3 \cdots n_k$.

This rule tells us that the number of ways of placing k balls in n bins with replacement is n^k.

The rule is more interesting when we are placing k balls in n bins without replacement. Now there are n ways of placing the first ball, and *no matter* where it was placed there are exactly $n-1$ bins in which the second ball may be placed (exactly which $n-1$ depends upon which bin the first ball was placed in), and so on. So as long as $k \leq n$, the number of placements is $n(n-1)\cdots(n-k+1) = \frac{n!}{k!}$. By convention we assume that $0! = 1$.

While dealing a hand of cards, say a poker hand, it is more natural to count the number of distinct hands (i.e. the set of 5 cards dealt in the hand), rather than the order in which they were dealt. To count this number we use the second rule of counting:

Second Rule of Counting: If an object is made by a succession of choices, and the order in which the choices is made does not matter, count the number of ordered objects (pretending that the order matters), and divide by the following number — the number of ordered objects per unordered object. Note that this rule can only be applied if the number of ordered objects is the same for every unordered object.

Let us continue with our example of a poker hand. We wish to calculate the number of ways of choosing 5 cards out of a deck of 52 cards. So we first count the number of ways of dealing a 5 card hand pretending that we care which order the cards are dealt in. This is exactly $\frac{52!}{47!}$ as we computed above. Now we ask for a given poker hand how many ways could it have been dealt? The 5 cards in the given hand could have been dealt in any one of 5! ways. Therefore by the second rule of counting, the number of poker hands is $\frac{52!}{47!5!}$.

This quantity $\frac{n!}{(n-k)!k!}$ is used so often that there is special notation for it: $\binom{n}{k}$, pronounced *n choose k*.