Problem Set 12

1. **Saving the Spotted Foobar**

   As you may've heard, there's an endangered population of the rare and elusive spotted foobars living in the tunnels deep under Evans Hall. They only venture outside to eat through the grate outside Evans, usually in the middle of the night.

   To comply with new safety laws, the campus plans to put in a new grating, which has 5-inch-wide holes. You've been hired by the Save the Spotted Foobar Coalition to ensure the new grating doesn't seriously impact the spotted foobars' feeding.

   According to the latest zoology studies, the spotted foobars in the current population are, on average, skinny enough to fit through a hole of diameter 1.9 in, with a variance of $2.3\,\text{in}^2$.

   (a) Out of the bounds we've shown in the CS70, use the strongest one you can to obtain a definite bound on these probabilities:

      i. The probability of a random spotted foobar fitting through the grating to go out to forage.

      ii. The probability that a random spotted foobar will drop through the new grating no matter how it orients itself, assuming that the zoologists have measured their longest dimension to be 3.1 inches on average, with a variance of $4.3\,\text{in}^2$. (After all, it's dark, and the spotted foobars might be trying to get back in in a rush if a predator appears.)

      iii. The probability that, out of the current population of 343 spotted foobars, no more than 20 starve to death due to not being able to get outside. (The spotted foobars don't share their food — and there are no vending machines in the steam tunnels.)

   (b) You appeal to PPCS to use a wider grating: since the spotted foobars are almost extinct as is, losing 20 more would be terrible. The people in charge tell you, "that's ridiculous, the previous grate had holes 2.4 inches wide, how could we make it worse by putting in one with wider holes?" Using the data given, can you definitively prove that they're lying about the size of the previous grate? Assume any spotted foobar that couldn't squeeze through the previous grate at all would've starved to death before the statistics above were collected.

2. **Messing with Chernoff**

   One night, word spreads in the 2nd floor labs that there are 4 dozen free donuts upstairs. The 25 people in 271 are working on a looming midnight 61C deadline, and are each 30% likely to stay put and ignore the donuts. The 40 people in the other labs aren't as stressed, but are each 10% likely to have already filled up on pizza (per midterm 2). Assume that everyone makes their decision independently.

   (a) Use the Chernoff bound from class to bound the probability that there'll be enough donuts for everyone (assuming, unrealistically, that no one takes seconds).

   (b) We derived the Chernoff bound by applying the Markov bound to $\alpha^{X_1+\cdots+X_n}$. What happens in this problem if you use the same procedure but start by applying the Markov bound to $2^{X_1+\cdots+X_n}$?

3. **To infinities, and beyond**

   Show whether each of the following sets is finite, countably infinite, or uncountable:

   (a) The set of all primes.

   (b) The set of all real-valued random variables on a finite sample space.

   (c) The set of all integer-valued random variables defined on the sample space Ω of positive integers, with $\Pr[\omega] = 1/2^\omega$

(d) The set of all integer-valued random variables on a finite sample space.

(e) The set of all possible functions from $\mathbb{Z}_{97}$ to $\mathbb{Z}_{97}$.

(f) The set of all graphs.