Hypercubes

Recall that the set of all \( n \)-bit strings is denoted by \( \{0,1\}^n \). The \( n \)-dimensional hypercube is a graph whose vertex set is \( \{0,1\}^n \) (i.e. there are exactly \( 2^n \) vertices, each labeled with a distinct \( n \)-bit string), and with an edge between vertices \( x \) and \( y \) iff \( x \) and \( y \) differ in exactly one bit position. i.e. if \( x = x_1x_2 \ldots x_n \) and \( y = y_1y_2 \ldots y_n \), then there is an edge between \( x \) and \( y \) iff there is an \( i \) such that \( \forall j \neq i, x_j = y_j \) and \( x_i \neq y_i \).

There is another equivalent recursive definition of the hypercube:

The \( n \)-dimensional hypercube consists of two copies of the \( n-1 \)-dimensional hypercube (the 0-subcube and the 1-subcube), and with edges between corresponding vertices in the two subcubes. i.e. there is an edge between vertex \( x \) in the 0-subcube (also denoted as vertex 0\( x \)) and vertex \( x \) in the 1-subcube.

Claim: The total number of edges in an \( n \)-dimensional hypercube is \( n2^{n-1} \).

Proof: Each vertex has \( n \) edges incident to it, since there are exactly \( n \) bit positions that can be toggled to get an edge. Since each edge is counted twice, once from each endpoint, this yields a grand total of \( n2^n/2 \).

Alternative Proof: By the second definition, it follows that \( E(n) = 2E(n-1) + 2^{n-1} \), and \( E(1) = 1 \). A straightforward induction shows that \( E(n) = n2^{n-1} \).

We will prove that the \( n \)-dimensional hypercube is a very robust graph in the following sense: consider how many edges must be cut to separate a subset \( S \) of vertices from the remaining vertices \( V - S \). Assume that \( S \) is the smaller piece; i.e. \( |S| \leq |V - S| \).

Theorem: \( |E_{S,V-S}| \geq |S| \).

Proof: By induction on \( n \). Base case \( n = 1 \) is trivial.

For the induction step, let \( S_0 \) be the vertices from the 0-subcube in \( S \), and \( S_1 \) be the vertices in \( S \) from the 1-subcube.

Case 1: If \( |S_0| \leq 2^{n-1}/2 \) and \( |S_1| \leq 2^{n-1}/2 \) then applying the induction hypothesis to each of the subcubes shows that the number of edges between \( S \) and \( V - S \) even without taking into consideration edges that cross between the 0-subcube and the 1-subcube, already exceed \( |S_0| + |S_1| = |S| \).

Case 2: Suppose \( |S_0| > 2^{n-1}/2 \). Then \( |S_1| \leq 2^{n-1}/2 \). But now \( |E_{S,V-S}| \geq 2^n - 1 \geq |S| \). This is because by the induction hypothesis, the number of edges in \( E_{S,V-S} \) within the 0-subcube is at least \( 2^{n-1} - |S_0| \), and those within the 1-subcube is at least \( |S_1| \). But now there must be at least \( |S_0| + |S_1| \) edges in \( E_{S,V-S} \) that cross between the two subcubes (since there are edges between every pair of corresponding vertices. This is a grand total of \( 2^{n-1} - |S_0| + |S_1| + |S_0| - |S_1| = 2^{n-1} \).
Hamiltonian Tours and Paths

A Hamiltonian tour in an undirected graph \( G = (V,E) \) is a cycle that \textit{goes through every vertex exactly once}. A Hamiltonian path is a path that goes through every vertex exactly once.

\textbf{Theorem:} For every \( n \geq 2 \), the \( n \)-dimensional hypercube has a Hamiltonian tour.

\textbf{Proof:} By induction on \( n \). In the base case \( n = 2 \), the 2-dimensional hypercube, the length four cycle starts from 00, goes through 01, 11, and 10, and returns to 00.

Suppose now that every \( (n - 1) \)-dimensional hypercube has an Hamiltonian cycle. Let \( v \in \{0,1\}^{n-1} \) be a vertex adjacent to \( 0^n \) (the notation \( 0^n \) means a sequence of \( n \) zeroes) in the Hamiltonian cycle in a \( (n - 1) \)-dimensional hypercube. The following is a Hamiltonian cycle in an \( n \)-dimensional hypercube: have a path that goes from \( 0^n \) to \( 0v \) by passing through all vertices of the form \( 0x \) (this is simply a copy of the Hamiltonian path in dimension \( n - 1 \)), minus the edge from \( v \) to \( 0^n \), then an edge from \( 0v \) to \( 1v \), then a path from \( 1v \) to \( 10^{n-1} \) that passes through all vertices of the form \( 1x \), and finally an edge from \( 10^{n-1} \) to \( 0^n \). This completes the proof of the Theorem.

When we start from \( 0^n \) and we follow the Hamiltonian tour described in the above proof, we find an ordering of all the \( n \)-bit binary strings such that each string in the sequence differs from the previous string in only one bit. Such an ordering is called a \textbf{Gray code} (from the name of the inventor) and it has various application.