CS 70 Spring 2007 — Discrete Mathematics for CS — Luca Trevisan — Lecture 21

Some Important Distributions

The binomial distribution

Let X be the number of Heads in n tosses of a biased coin with Heads probability p. Clearly X takes on the values $0, 1, \ldots, n$. As we saw in an earlier lecture, its distribution is

$$\Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}. \tag{1}$$

Definition 21.1 (binomial distribution): A random variable having the distribution (1) is said to have the binomial distribution with parameters n and p.

Recall from Lecture Notes 19 that the expectation of a binomial random variable is $\mathrm{E}(X) = np$. A plot of the binomial distribution (when n is large enough) looks more-or-less bell-shaped, with a sharp peak around the expected value np.

We computed in Lecture 20 that $\mathrm{Var}(X) = n \cdot (p - p^2)$, and so the standard deviation is $\sqrt{n \cdot (p - p^2)}$.

The Poisson distribution

We shall now see that the binomial distribution has a nice approximation in the case where the average pn is small and n is very large. Such distributions arise when there are many agents that independently decide whether or not to perform some action, and the action is performed with small probability. This model approximates several real-life situations such as traffic patterns, volume of calls in call centers, etc.

So say that we have n people, each one, on a given day, may or may not call the call center, their actions are independent, and we know that average number of people calling on a given day is a certain value λ. Let X be the random variable denoting the number of calls in a day; then X has the binomial distribution with parameter $p = \frac{\lambda}{n}$.

Throw n balls into $\frac{n}{\lambda}$ bins (where λ is a constant). Let X be the number of balls that land in bin 1. Then X has the binomial distribution with parameters n and $p = \frac{\lambda}{n}$, and its expectation is $\mathrm{E}(X) = np = \lambda$. (Why?)

Let's look in more detail at the distribution of X. Beginning with $p\Pr[X = 0]$, we have

$$\Pr[X = 0] = \Pr[\text{no person calls}] = \left(1 - \frac{\lambda}{n}\right)^n \to \mathrm{e}^{-\lambda} \quad \text{as } n \to \infty.$$

So the probability of no person calling will be very close to the constant value $\mathrm{e}^{-\lambda}$ when n is large.

[We use that fact, which you may remember from your calculus class, that for every constant c we have $\lim_{n\to\infty} \left(1 - \frac{c}{n}\right)^n = e^c$.]

What about $\Pr[X = i]$ for $i > 0$? Well, we know from the binomial distribution that $\Pr[X = i] = \binom{n}{i}(\frac{\lambda}{n})^i(1 - \frac{\lambda}{n})^{n-i}$. In particular,

$$\begin{aligned}
\Pr[X = 1] &= n \cdot \frac{\lambda}{n} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-1} \\
&= \lambda \cdot \frac{1}{\left(1 - \frac{\lambda}{n}\right)} \cdot \left(1 - \frac{\lambda}{n}\right)^{n} \\
&\to \lambda \cdot e^{-\lambda} \quad \text{as } n \to \infty.
\end{aligned}$$

In general we can use a similar trick to write, for each i, the expression for the probability in a form that makes it easy to take the limit $n \to \infty$.

$$\begin{aligned}
\Pr[X = i] &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \cdot \frac{1}{\left(1 - \frac{\lambda}{n}\right)^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n} \\
&\to \frac{\lambda^k}{k!} \cdot e^{-\lambda} \quad \text{as } n \to \infty.
\end{aligned}$$

This motivates the following definition:

Definition 21.2 (Poisson distribution): A random variable X for which

$$\Pr[X = i] = \frac{\lambda^i}{i!}\mathrm{e}^{-\lambda} \qquad \text{for } i = 0, 1, 2, \ldots \tag{2}$$

is said to have the <u>Poisson distribution with parameter λ</u>.

To make sure this definition is valid, we had better check that (2) is in fact a distribution, i.e., that the probabilities sum to 1. We have

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}\mathrm{e}^{-\lambda} = \mathrm{e}^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = \mathrm{e}^{-\lambda} \times \mathrm{e}^{\lambda} = 1.$$

[In the second-last step here, we used the Taylor series expansion $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.]

What is the expectation of a Poisson random variable X? This is a simple hands-on calculation, starting from the definition of expectation:

$$\begin{aligned}
\mathrm{E}(X) &= \textstyle\sum_{i=0}^{\infty} i \times \Pr[X=i] \\
&= \textstyle\sum_{i=0}^{\infty} i \frac{\lambda^i}{i!} \mathrm{e}^{-\lambda} \\
&= \mathrm{e}^{-\lambda} \textstyle\sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} \\
&= \lambda \mathrm{e}^{-\lambda} \textstyle\sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda \mathrm{e}^{-\lambda} \mathrm{e}^{\lambda} \\
&= \lambda.
\end{aligned}$$

So <u>the expectation of a Poisson r.v. X with parameter λ is $\mathrm{E}(X) = \lambda$</u>.

To compute the variance, we have to compute the expectation of the random variable X^2.

$$\mathrm{E}(X^2) = \sum_{i=0}^{\infty} i^2 \mathrm{e}^{-\lambda} \frac{\lambda^i}{i!} = \lambda \sum_{i=1}^{\infty} i \mathrm{e}^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!} = \lambda \left(\sum_{i=1}^{\infty} (i-1) \mathrm{e}^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!} + \sum_{i=1}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!} \right) = \lambda(\lambda+1).$$

[Check you follow each of these steps. In the last step, we have noted that the two sums are respectively $\mathrm{E}(X)$ and $\sum_i \Pr[X=i] = 1$.]

Finally, we get $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \lambda$. So, for a Poisson random variable, the expectation and variance are equal.

A plot of the Poisson distribution reveals a curve that rises monotonically to a single peak and then decreases monotonically. The peak is as close as possible to the expected value, i.e., at $i = \lfloor \lambda \rfloor$.

We have seen that the Poisson distribution arises as the limit of the number of balls in bin 1 when n balls are thrown into $\frac{n}{\lambda}$ bins. In other words, it is the limit of the binomial distribution with parameters n and $p = \frac{\lambda}{n}$ as $n \to \infty$, with λ being a fixed constant. The Poisson distribution is also a very widely accepted model for so-called "rare events", such as misconnected phone calls, radioactive emissions, crossovers in chromosomes, etc. This model is appropriate whenever the events can be assumed to occur randomly with some constant density λ in a continuous region (of time or space), such that events in disjoint subregions are independent. One can then show that the number of events occurring in a region of unit size should obey the Poisson distribution with parameter λ.

Here is a slightly frivolous example. Suppose cookies are made out of a dough that contains (on average) three raisins per spoonful. Each cookie contains two spoonfuls of dough. Then we would expect that, to good approximation, the number of raisins in a cookie is has the Poisson distribution with parameter $\lambda = 6$. Here are the first few values:

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\Pr[X=i]$	0.002	0.015	0.045	0.089	0.134	0.161	0.161	0.138	0.103	0.069	0.041	0.023	0.011

Notice that the Poisson distribution arises naturally in (at least) two distinct important contexts. Along with the binomial and the normal distributions (which we shall meet soon), the Poisson distribution is one of the three distributions you are most likely to find yourself working with.