Practice problems 2

Only if you don't have time: solve or at least attempt the problems following the corresponding directions.

ALL of the following problems:

1. Prove or disprove that $n^5 - n$ is divisible by 5, whenever n is a nonnegative integer.

2. Describe a method to share a secret with b bits with n people such that each person is given at most $b/k+1$ bits and any k of the people can reconstruct all of the bits, and any $k-1$ people won't know anything about the first b/k bits of the secret. (Use a Galois Field, $GF(p)$ for your field, and state how big p should be.)

ALL of the following problems:

1. Show that any graph where every node has degree larger than $n/2$ has a Hamiltonian cycle.

2. A *simple k-cycle* in a simple graph is an undirected path going through each of k distinct vertices exactly once and ending where it started (where $k > 2$). A simple k-cycle can be represented by the sequence of k distinct vertices $v_1, v_2, \ldots, v_k$ along the path.

   Note that every simple k-cycle can be represented by many sequences. For example, the 4-cycle represented by 1234 is as the same cycle represented by 2341 or 3412 because a cycle does not have to start at any particular vertex. It is also represented by 4321, because the cycle is undirected

   (a) How many different sequences of vertices represent a given simple k-cycle?

   (b) In a complete graph on n vertices (i.e. all possible edges are present), how many simple k-cycles are there?
   Suppose we construct a simple n-node graph, $G = (V, E)$, randomly as follows: For every set of two distinct vertices v, u, toss a biased coin whose probability of coming up heads is p. The undirected edge between v and u. is included in E iff the coin comes up heads. Assume that all coin tosses are mutually independent.

   (c) What is the expected number of simple k-cycles in G?

The following problem:

1. Prove that
$$\sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n.$$
(Hint: consider the number of ways of splitting n elements into 3 groups.)

At least two of the following problems:

1. There are 4 different coins in a box. The probability of Heads when flipping the ith coin is $1/i$ for $1 \le i \le 4$. A coin is selected from the box randomly, and gets tossed until a Head appears.

(a) Write down a probability space for the experiment. Be sure to verify that the sum of the sum of the probabilities of the sample point is 1.

(b) What is the probability that a Head is first seen in the 2nd toss?

(c) Given that a Head is first seen in the 2nd toss, what is the probability that ith coin was selected from the box.

(d) Check your answer to part (c) to be sure it satisfies:

$$\Pr(C_1 \mid H_2) + \Pr(C_2 \mid H_2) + \Pr(C_3 \mid H_2) + \Pr(C_4 \mid H_2) = 1$$

2. A box contains tickets numbers $1, 2, \ldots, N$. m tickets are drawn with replacement. What is the probability that the largest number drawn is k?

3. I write a letter to my friend and don't receive a reply. One out of m letters gets lost in the mail. What is the probability my friend received my letter, assuming that if he did, he would reply?

At least two of the following problems:

1. Give a distribution for a random variable where the expectation is $1/2$ and the probability that it is greater than or equal to 1 is $2/3$.

2. In an equivalent class at Stanford, the staff decides to collect the students' cell phones before the exam. However, since they were unable to come up with a systematic way of collecting and redistributing the cell phones, they just decide to distribute back the cell phones randomly. Suppose they are n students in that class and each owns a cell phone.

   (a) What is the expected number of students who get their own cell phones?

   (b) What is the expected number of pairs of students who'll need to swap cell phones?

   (c) What is the expected number of k-tuples of students who can stand in a circle and each hand their cell phones to the person on their left?

   (d) What is the expected total number of circles that will form this way (a student who has got his own cell phone is a circle of length 1, a pair of students who swap cell phones are a circle of size 2, etc.)?

3. Suppose that you are given two biased coins C_p and C_r . Coin C_p flips a head with probability p and coin C_r flips a head with probability r.
   The game is to flip C_p and then C_r and keep alternating between coins until you get 2l heads in a row, where l is a given positive integer. However, as soon as you get a tail, then you start over again by first flipping C_p and then alternating between coins. Assume that coin flips are mutually independent of each other.

   (a) For $l = 3$, write an expression in p and r for the the expected number of coin flips to see $2l$ heads in a row.

   (b) What is the expected number of coin flips to see $2l$ heads in a row?
   Note: You may express your answer using summations determined by l, but if you do, you should briefly indicate how results from the course imply that there are closed forms for your summations.

The following problem:

1. The covariance, $Cov(X, Y)$, of two random variables, X and Y , is defined to be $E(XY) - E(X)E(Y)$. Note that if two random variables are independent, then their covariance is zero.

   (a) Give an example to show that having $Cov(X, Y) = 0$ does not necessarily mean that X and Y are independent.

   (b) Let $X_1, \ldots, X_n$ be random variables. Prove that

$$Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$