EECS 70 Discrete Mathematics and Probability Theory
Spring 2014 Anant Sahai

Discussion 11B

1. **Throwing balls into a depth-limited bin**

   Say you want to throw n balls into n bins with depth $k-1$ (they can fit $k-1$ balls, after that the bins overflow). Suppose that n is a large number and $k = 0.1n$. You throw the balls randomly into the bins, but you would like it if they don't overflow. You feel that you might expect not too many balls to land in each bin, but you're not sure, so you decide to investigate the probability of a bin overflowing.

   (a) Focus on the first bin. Get an upper bound the number of ways that you can throw the balls into the bins such that this bin overflows. Try giving an argument about the following strategy: select k balls to put in the first bin, and then throw the remaining balls randomly.

   (b) Calculate an upper bound on the probability that the first bin will overflow:

   (c) Upper bound the probability that some bin will overflow.

   (d) How does the above probability scale as n gets really large?

2. **Coin tosses, revisited**

   You may want to use a calculator to find binomial coefficients for this problem. Or you could use a computer. For instance, in Python, if you have the `scipy` module installed, you can find $\binom{n}{k}$ using `scipy.misc.comb(n,k,exact=1)`. Or you could type things like `100 choose 50` into Wolfram Alpha (`https://www.wolframalpha.com`).

   Suppose you toss a fair coin $N = 10$ times. You would expect to see roughly 5 heads, right? Let's explore this in some depth.

   (a) What are the chances that you will see *exactly* 5 heads when you toss 10 fair coins? (As a related question to help you think about this, if you had the opportunity to pay \$1 to play a game where you earned \$1.25 if 10 coin tosses resulted in exactly 5 heads, would you play that game?)

   (b) Now let $N = 20$, 50, and 100. In all these cases, what are the chances that you will see exactly $N/2$ heads when you toss N fair coins? Are these chances increasing or decreasing as N increases? Do you find this surprising, given what you expect?

   (c) Let's see if including a margin of error changes anything. Suppose you toss N coins (where N is even). What are the chances that you will see either $\frac{N}{2}$, or $\frac{N}{2}-1$, or $\frac{N}{2}+1$ heads, for $N = 10$, 20, 50, and 100. Are these chances increasing or decreasing as N increases? Do you find this surprising, given what you expect?

   (d) Let's see if including a *percentage* margin of error changes anything. Suppose you toss N coins (where N is divisible by 5). What are the chances that you will see anywhere between 40% and 60% of your tosses returning heads, for $N = 10$, 20, 50, and 100. Are these chances increasing or decreasing as N increases? Do you find this surprising, given what you expect?

3. **Hashing**

(a) Given k items, how big a hashtable do we need so that when the k items are hashed, there are no collisions with at least constant probability, say p? Assume the hash function assigns items to buckets independently and uniformly at random.

For this part, take p to be large enough to force n to be much larger than k.

You might need to use $\log(1-x) \approx -x$ for small x.

(b) Given k items, how big a hashtable do we need so that when the k items are hashed, there are no collisions with *a particular item*, call it x, with constant probability? Same assumptions on the hash function as above.