1. **Shading areas in probability**

Let's say you toss a coin 1000 times, and you calculate the fraction of heads. The central limit theorem tells you that the distribution of the fraction of heads looks approximately like that of a Gaussian random variable. In simpler terms, the shape of the distribution will look like a bell centered at 0.5, and rapidly falling on either side.

(a) Sketch the bell curve above in free hand. You don't have to be too accurate; just get the general shape right.
(b) Shade the area under the curve that represents the outcome of getting more than 800 heads.
(c) Shade the area under the curve that represents the outcome of getting less than 200 heads.
(d) Shade the area under the curve that represents the outcome of getting no more than 50 heads away from the average.
(e) Shade the area under the curve that represents the outcome of getting more than 200 heads away from the average.

2. **Erasures, Bounds, and Probabilities**

You may want to use a calculator for this problem. Better yet, use a computer running MATLAB or Python to calculate things like erf functions. Or you can just type stuff like `erf(0.25)` into Wolfram Alpha (`http://www.wolframalpha.com`). Or you can use the table on the next page.

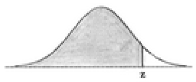
Alice is sending 1000 bits to Bob. The probability that a bit gets erased is p, and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate upto one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most 10^{-6}.

(a) Use Markov's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.
(b) Use Chebyshev's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}. Recall that the variance of the number of erased bits will be $1000p(1-p)$, as proved in a previous discussion section.
(c) Use the following Chernoff bound (which is valid for all $\varepsilon > 0$) to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

$$\Pr\left(\text{No. of erasures } \geq (1+\varepsilon)1000p\right) \leq e^{-\frac{\varepsilon^2}{2+\varepsilon}1000p} \quad \text{(Chernoff Bound)}$$



Probability Content from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

(d) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable[1] (with suitable mean and variance). Use this to find an approximate bound for p such that the probability of a communications breakdown is at most 10^{-6}. You can use the fact that if Y is a Gaussian random variable with mean μ and variance σ^2, then:

$$\Pr\left(Y \geq y\right) = \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{y-\mu}{\sigma\sqrt{2}}\right)\right)$$

[1] A Gaussian random variable is an example of a continuous random variable, which you have not seen too much of in this course. Basically, the random variable can be any real number, instead of being confined to take only one of a discrete set of values. For continuous random variables like the Gaussian random variable, it does not make much sense to ask "what is the probability of the variable assuming a particular value, like 0 or 0.5" (since this probability is always 0 for any such single point). Instead, one asks "what is the probability that the variable takes a value in a given range, like for example, $[-1,1)$ or $(5,\infty)$. What the CLT says is that when you average a large number of discrete i.i.d Bernoulli random variables, even though the average is still a discrete random variable, the distribution of the average begins to look more and more like that of a continuous (Gaussian) random variable.