EECS 70 Discrete Mathematics and Probability Theory
Spring 2014 Anant Sahai

Discussion 4A

1. **Strategic Women**

   Assume that you are running the stable marriage algorithm with men proposing. We learned in class that this algorithm results in a pairing that is pessimal for women. Part of the reason behind this is that women do not act strategically.

   Try to construct an example of stable marriage (with as many men and women as you want) in which the following scenario happens: the algorithm ends (i.e. results in a pairing) with woman A being paired with man 1. After the algorithm ends, woman A goes to see her woman friend B who is paired with man 2. They both realize that they prefer being matched to each other's matches. Meaning that A would prefer 2 over 1 and B would prefer 1 over 2. Can your example be such that woman A can realize a better matching for herself and her friend B by lying at some point about her preferences? Hint: Her lie can result in her rejecting a proposal that she normally would not reject.

   Now construct an example for every $n \geq 2$ where a single woman can strategically reject a proposal to make sure that every woman gets a strictly better choice at the end of the day (compared to what they would have gotten if they only followed their preferences and did not act strategically).

2. **Marriage is not for everyone**

   Believe it or not, there are people who prefer being single over being married to certain people. In their preference lists they have a spot for being single (which may lie anywhere on the list).

   - How should we change the meaning of rogue couples? What would not be a stable pairing? (Remember that a pairing no longer has to cover every person. You are trying to model real-life scenarios, i.e. a married person who prefers being single can simply get separated from his/her partner and be a rogue element that destabilizes the pairing).
   - With this new definition of stable pairing, how can we prove that a stable pairing always exists, and how can we find one such stable pairing? (Hint: try to reduce to the case where there is no choice of being single.)
   - Do stable pairings change if we change the order of people below the single choice on the preference lists?

3. **Modular decomposition of modular arithmetic**

   Complex systems are always broken down into simpler modules. In this problem you will learn how this might be done in modular arithmetic.

   (a) Write down the addition and multiplication table for modular-6 arithmetic (the rows and columns should be labeled $0,1,2,3,4,5$).

   (b) Each number $0,1,2,3,4,5$ has a remainder mod 2 and a remainder mod 3. For each number write down the pair (x,y) where x is its remainder mod 2 and y is its remainder mod 3. Obviously $0 \leq x \leq 1$ and $0 \leq y \leq 2$. Out of all possible pairs (x,y), where $0 \leq x \leq 1$ and $0 \leq y \leq 2$, how many times do you see each pair appear?

(c) Again write down the addition and multiplication table you wrote in part 1, but this time replace each number with its corresponding pair (when a number appears as a row/column label and also when it appears somewhere in the table). Describe how one can add or multiply two pairs without looking at the original numbers.

4. **Proofs (or counterexamples) for dividing numbers**

   (a) Prove that if a prime number p divides $a \cdot b$, then p divides a or p divides b.

   (b) If an integer m divides both a and b, prove that m divides $x \cdot a + y \cdot b$ for any integers x, y.

   (c) If m and n both divide a, does this mean mn divides a?

   (d) Now, assume m and n are prime. If m and n both divide a, does this mean mn divides a?