1. Sanity Check!

a. Define $X$ to be the sum of $n$ standard six-sided dice. What is $E[X]$?

b. Suppose we have a biased coin that comes up heads with probability $p$. After $n$ tosses, what is the expected number of occurrences of the subsequence $HTH$? (For example, the sequence $HTHHTHTTH$ has two occurrences of $HTH$.)

2. Bernoulli and Binomial Distribution

A random variable $X$ is called a Bernoulli random variable with parameter $p$ if $X = 1$ with probability $p$ and $X = 0$ with probability $1 - p$.

a. Calculate $E[X]$ and $\text{Var}[X]$.

b. A Binomial random variable with parameters $n$ and $p$ is defined to be the sum of $n$ independent, identically distributed Bernoulli random variables with parameter $p$. If $Z$ is a Binomial random variable with parameters $n$ and $p$, what are $E[Z]$ and $\text{Var}[Z]$?

3. Chopping up DNA

In a certain biological experiment, a piece of DNA consisting of a linear sequence (or string) of 4000 nucleotides is subjected to bombardment by various enzymes. The effect of the bombardment is to randomly cut the string between pairs of adjacent nucleotides: each of the 3999 possible cuts occurs independently and with probability $1/500$.

a. What is the expected number of pieces into which the string is cut?
b. What is the variance of the above quantity? (Hint: use problem 2.)

c. Suppose that the cuts are no longer independent, but highly correlated: when a cut occurs in a particular location, nearby locations are much more likely to be cut as well. The probability of each individual cut remains 1/500. Does the expected number of pieces increase, decrease, or stay the same?

4. Will I Get My Package?
A sneaky delivery guy of some company is out delivering \( n \) packages to \( n \) customers. Not only does he hand a random package to each customer, he tends to open a package before delivering with probability \( \frac{1}{2} \) (independently of the choice of the package). Let \( X \) be the number of customers who receive their own packages unopened.

a. Compute the expectation \( \text{E}(X) \).

b. What is the probability that customers \( i \) and \( j \) both receive their own packages unopened?

c. Compute the variance \( \text{Var}(X) \).