CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand Discussion 2B

1. Stable Marriage

Consider the following list of preferences:

Men	Preferences
A	$4 > 2 > 1 > 3$
B	$2 > 4 > 3 > 1$
C	$4 > 3 > 1 > 2$
D	$3 > 1 > 4 > 2$

Women	Preferences
1	$A > D > B > C$
2	$D > C > A > B$
3	$C > D > B > A$
4	$B > C > A > D$

1. Is $\{(A,4),(B,2),(C,1),(D,3)\}$ a stable pairing?
2. Find a stable matching by running the Traditional Propose & Reject algorithm.
3. Show that there exist a stable matching where women 1 is matched to men A.

2. True or False

For each of the following statements about the traditional stable marriage algorithm with men proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

1. (True/False) In a stable marriage algorithm execution which takes n days, there is a woman who did not receive a proposal on the $(n-1)$th day.
2. (True/False) In a stable marriage algorithm execution, if a woman receives a proposal on day k, she receives a proposal on every subsequent day until termination.
3. (True/False) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every man ends up with his least preferred woman.

4. (True/False) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every woman ends up with her least preferred man.

5. (True/False) In a stable marriage algorithm execution, if woman W receives no proposal on day i, then she receives no proposal on any previous day j which is less than i.

3. Large Number of Stable Pairings

How many different stable pairings can there be for an instance with n men and n women? In this question we will see how to construct instances with a very large number of stable pairings.

The overall plan is as follows: imagine we have already constructed an instance of stable marriage with m men and m women which admits X different stable pairings. We will show how to use this to construct a new instance with $2m$ men and $2m$ women which has at least X^2 stable pairings.

1. Construct an instance with $n = 2$ and 2 different stable pairings. Now assuming construction in the overall plan above, show that there is an instance with $n = 4$ and 4 different stable pairings. What does that tell you about $n = 8$? In general if we continue this for $n = 2^k$, how many different stable pairings do we get? Express that as a function of n.

2. Implement the overall plan: square the number of stable pairings at the cost of doubling the size of the instance. Start with an instance of stable marriage with m men and m women which admits X different stable pairings, and create a new instance with $2m$ men and $2m$ women. To do this, create two copies of each person in the instance you start with. Call one of them the original, and the other the alternate. Let original people prefer original people above alternate people, and alternate people prefer alternate people above originals, but in every other way let preferences remain as they were in the instance you started with. Thus, for example, the preference list of an alternate man would look like the preference list repeated twice of the man he was cloned from in the original instance, with the first half consisting of alternate women and the second half consisting of original women. Prove that the number of stable

pairings in this new instance is at least X^2. To do so, it suffices to exhibit that for any pair of stable pairings of the instance you started with there is a unique stable pairing in the new instance.

For home Good, Better, Best

In a particular instance of the stable marriage problem with n men and n women, it turns out that there are exactly three distinct stable matchings, M_1, M_2, and M_3. Also, each man m has a different partner in the three matchings. Therefore each man has a clear preference ordering of the three matchings (according to the ranking of his partners in his preference list). Now, suppose for man m_1, this order is $M_1 > M_2 > M_3$.

Prove that every man has the same preference ordering $M_1 > M_2 > M_3$.