1. Set Operations

- $\mathbb{N}$ denotes the set of all natural numbers: $\{0,1,2,3,...\}$.
- $\mathbb{Z}$ denotes the set of all integer numbers: $\{\ldots,-2,-1,0,1,2,\ldots\}$.
- $\mathbb{Q}$ denotes the set of all rational numbers: $\{\frac{a}{b} \mid a,b \in \mathbb{Z},\, b \neq 0\}$.
- $\mathbb{R}$ denotes the set of all real numbers.

(a) $\mathbb{N} \cap \mathbb{Z} = \mathbb{N}$.

(b) $\mathbb{Z} \cap \mathbb{Q} = \mathbb{Z}$.

(c) $\mathbb{Q} \cap \mathbb{R} = \mathbb{Q}$.

(d) $\mathbb{N} \cup \mathbb{Z} = \mathbb{Z}$.

(e) $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$.

(f) $\mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

(g) $\mathbb{N} \setminus \mathbb{Z} = \emptyset$.

(h) $\mathbb{Z} \setminus \mathbb{N} =$ the set of negative integers.

(i) $\mathbb{Z} \setminus \mathbb{Q} = \emptyset$.

(j) $\mathbb{Q} \setminus \mathbb{Z} =$ the set of non-integer rational numbers.

(k) $\mathbb{Q} \setminus \mathbb{R} = \emptyset$.

(l) $\mathbb{R} \setminus \mathbb{Q} =$ the set of irrational numbers.

2. Summation and Product

(a) $\sum_{i=1}^{3} i = 1+2+3 = 6$.

(b) $\prod_{i=1}^{3} i = 1 \times 2 \times 3 = 6$.

(c) $\sum_{i=1}^{3} \sum_{j=1}^{3} j = (1+2+3)+(1+2+3)+(1+2+3) = 18$.

(d) $\prod_{i=1}^{3} \sum_{j=1}^{i} j = (1) \times (1+2) \times (1+2+3) = 18$.

3. (Truth table) Use truth tables to show that $\neg(A \vee B) \equiv \neg A \wedge \neg B$ and $\neg(A \wedge B) \equiv \neg A \vee \neg B$. These two equivalences are known as DeMorgan's Law.

A	B	$\neg(A \vee B)$	$\neg A \wedge \neg B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
0	0	1	1	1	1
0	1	0	0	1	1
1	0	0	0	1	1
1	1	0	0	0	0

4. (Writing in propositional logic.)

For each of the following sentences, translate the sentence into propositional logic using the notation introduced in class, and write its negation.

1. The square of a nonzero integer is positive.
2. There are no integer solutions to the equation $x^2 - y^2 = 10$.
3. There is one and only one real solution to the equation $x^3 + x + 1 = 0$.
4. For any two distinct real numbers, we can find a rational number in between them.

1. We can rephrase the sentence as "if n is a nonzero integer, then $n^2 > 0$", which can be written as
$$\forall n \in Z, (n \neq 0) \rightarrow (n^2 > 0)$$
or equivalently as
$$\forall n \in Z, (n = 0) \vee (n^2 > 0).$$
The latter is easier to negate, and its negation is given by
$$\exists n \in Z, (n \neq 0) \wedge (n^2 \leq 0)$$
2. The sentence is
$$\forall x, y \in Z, x^2 - y^2 \neq 10.$$
The negation is
$$\exists x, y \in Z, x^2 - y^2 = 10$$
3. Let $p(x) = x^3 + x + 1$. The sentence can be read "there is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x." Or,
$$\exists x \in \mathbb{R}, (p(x) = 0) \wedge (\forall y in \mathbb{R}, (p(y) = 0) \implies (x = y)).$$
Its negation is given by
$$\forall x \in \mathbb{R}, (p(x) \neq 0) \vee (\exists y \in \mathbb{R}, (p(y) = 0) \wedge (x \neq y)).$$

4. The sentence can be read "if x and y are distinct real numbers, then there is a rational number z between x and y." Or,

$$\forall x, y \in \mathbb{R}, (x \neq y) \implies (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

Equivalently,

$$\forall x, y \in \mathbb{R}, (x = y) \vee (\exists z \in \mathbb{Q}, (x < z < y) \vee (y < z < x)).$$

The negation is

$$\exists x, y \in \mathbb{R}, (x \neq y) \wedge (\forall z \in \mathbb{Q}, ((z \leq x) \vee (z \geq y)) \wedge ((y \geq z) \vee (x \leq z))).$$

5. (Implication) Which of the following implications are true? Give a counterexample for each false assertion.

1. $\forall x, \forall y, P(x,y) \implies \forall y, \forall x, P(x,y)$.

   True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

2. $\exists x, \exists y, P(x,y) \implies \exists y, \exists x, P(x,y)$. True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

3. $\forall x, \exists y, P(x,y) \implies \exists y, \forall x, P(x,y)$. False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedant is true and the consequence is false, those the entire implication statement is false.

4. $\exists x, \forall y, P(x,y) \implies \forall y, \exists x, P(x,y)$. True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note that the two statements are not equivalent as the converse of this is statement 3, which is false.