CS 70	Discrete Mathematics and Probability Theory	
Spring 2016	Rao and Walrand	Discussion 2A Sol

1. **Power Inequality**

   Use induction to prove that for all integers $n \geq 1$, $2^n + 3^n \leq 5^n$.

   We use induction on n. The base case $n = 1$ is true because $2+3 = 5$. Assume the inequality holds for some $n \geq 1$. For $n+1$, we can write:

   $$2^{n+1} + 3^{n+1} \;=\; 2 \cdot 2^n + 3 \cdot 3^n \;<\; 3 \cdot 2^n + 3 \cdot 3^n \;=\; 3(2^n + 3^n) \overset{(*)}{\leq} 3 \cdot 5^n < 5 \cdot 5^n = 5^{n+1},$$

   where the inequality (*) follows from the induction hypothesis. This completes the induction.

2. **Triangle Inequality**

   Recall the triangle inequality, which states that for real numbers x_1 and x_2,

   $$|x_1 + x_2| \leq |x_1| + |x_2|.$$

   Use induction to prove the generalized triangle inequality:

   $$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

   We use induction on $n \geq 2$. The base case $n = 2$ is the usual triangle inequality. Assume the inequality holds for some $n \geq 2$ (this is the inductive hypothesis). For $n+1$, we can write:

   $$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| && \text{(by the usual triangle inequality)} \\ &\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}| && \text{(by the induction hypothesis).} \end{aligned}$$

   This completes the induction.

3. **(Induction) Prove that, for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.**

   - Base case: when $n = 1$, $\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2 \cdot 1+1)}{6}$.
   - Inductive hypothesis: assume for $n = k \geq 1$ that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.

- Inductive step:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1)+6(k+1))}{6} \\
&= \frac{(k+1)(2k^2+k+6k+6)}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

By the principle of induction, the claim is proved.

4. Convergence of Series

Use induction to prove that for all integers $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2.$$

Hint: Strengthen the induction hypothesis to $\sum_{k=1}^{n} \frac{1}{3k^{3/2}} \leq 2 - \frac{1}{\sqrt{n}}$.

We use induction on n. The base case $n = 1$ is true because $1/3 < 1$. Assume the inequality holds for some $n \geq 1$. For $n+1$, by the inductive hypothesis, we have that

$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}}.$$

Thus, to prove our claim, it suffices to show that

$$-\frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \leq -\frac{1}{\sqrt{n+1}}. \tag{1}$$

This is a purely arithmetic problem and there are multiple ways to proceed.

Notice that to prove the inequality (1), it suffices to show that

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq \frac{1}{3(n+1)^{3/2}} = \frac{1}{3(n+1)\sqrt{n+1}},$$

which is equivalent to showing that

$$\frac{\sqrt{n+1}}{\sqrt{n}} - 1 = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \geq \frac{1}{3(n+1)}.$$

So we want to show
$$\frac{\sqrt{n+1}}{\sqrt{n}} \geq \frac{1}{3(n+1)} + 1 = \frac{3n+4}{3n+3},$$
and squaring both sides means this is equivalent to
$$\frac{n+1}{n} \geq \frac{(3n+4)^2}{(3n+3)^2}.$$
At this point we cross-multiply, so we just need to show that
$$(n+1)(3n+3)^2 \geq n(3n+4)^2.$$
This is something that can be easily seen by expanding both sides and canceling terms, so we have shown Equation (1).

This computation allows us to conclude that
$$\sum_{k=1}^{n+1} \frac{1}{3k^{3/2}} = \sum_{k=1}^{n} \frac{1}{3k^{3/2}} + \frac{1}{3(n+1)^{3/2}} \leq 2 - \frac{1}{\sqrt{n}} + \frac{1}{3(n+1)^{3/2}} \overset{(1)}{\leq} 2 - \frac{1}{\sqrt{n+1}},$$
where we have used equation (1) for the last inequality. This concludes the induction.

5. Fibonacci: for home.

Recall, the Fibonacci numbers, defined recursively as $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$. Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

First, we should prove that all the fibonacci numbers are integer by induction: $P(k)$ is "F_k is an integer." This follows from the fact that F_1 and F_2 are integer, and the induction step follows from $F_k = F_{k-1} + F_{k-2}$, the (strong) induction hypothesis that F_{k-1} and F_{k-2} are integers and the fact that the integers are closed under addition.

Now we prove that for all natural numbers $k \geq 1$, F_{3k} is even. The base case, $k = 1$, is that $F_3 = 2$ is even, which is clear.

For the induction step, we have that $F_n = F_{n-1} + F_{n-2} = 2F_{n-2} + F_{n-3}$. Or that $F_{3k+3} = 2F_{3k+1} + F_{3k}$.

By the induction hypothese $F_{3k} = 2q$ for some q, and we have that $F_{3k+3} = 2(F_{3k+1} + q)$, which implies that it is even. Thus, by induction we have that all F_{3k} are even.