CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand Discussion 3B

1. Tournament

A *tournament* is defined to be a directed graph such that for every pair of distinct nodes v and w, exactly one of (v,w) and (w,v) is an edge (representing which player beat the other in a round-robin tournament). Prove that every tournament has a Hamiltonian path. In other words, you can always arrange the players in a line so that each player beats the next player in the line.

Answer:

We provide two possible answers: one using simple induction, and the other using strong induction.

Answer 1: We will prove this with induction on the number of nodes/players.

Base Case $n = 1$*:* There is only one player, so the claim is trivially true.

Inductive Hypothesis: Suppose for some $n \geq 1$, we can find a Hamiltonian path in a tournament of n players.

Inductive Step: Consider a tournament of $n+1$ players. Arbitrarily pick one player p_{n+1} to "hold out." From our inductive hypothesis, we can arrange the remaining n players in a line, say $p_1, p_2, \ldots, p_n$, such that p_i beat p_{i+1} for $1 \leq i \leq n-1$.

Let p_a be the last player that beat p_{n+1}. If there is no such p_a (i.e., p_{n+1} beat everyone), then we can place p_{n+1} before p_1, and we are done. Otherwise, reorder the players as follows:

$$p_1, p_2, \ldots, p_a, p_{n+1}, p_{a+1}, \ldots, p_n.$$

We know that p_{n+1} must have beaten p_{a+1} by definition (or else p_{a+1} would be the last player that beat p_{n+1}). If it turns out that $a = n$, we simply place p_{n+1} after p_n, and we still have a valid Hamiltonian path.

Therefore, for all $n \geq 1$, there exists a Hamiltonian path in a tournament of n players. $\square$

Answer 2: We will prove this with strong induction on the number of nodes/players.

Base Case $n = 1$*:* There is only one player, so the claim is trivially true.

Inductive Hypothesis: Suppose for all $1 \leq k < n$, we can find a Hamiltonian path in a tournament of k players.

Inductive Step: Consider a tournament of n players.

Arbitrarily pick one player p to "hold out." Let S be the set of players who beat p, and let T be the set of players who p beat. From our inductive hypothesis, we can find a Hamiltonian path in S and in T. Finally, to obtain a Hamiltonian path on all n players, we connect the last person in S to p, and p to the first person in T.

Therefore, for all $n \geq 1$, there exists a Hamiltonian path in a tournament of n players. $\square$

2. Leaves in a tree

A *leaf* in a tree is a vertex with degree 1.

(a) Prove that every tree on $n \geq 2$ vertices has at least two leaves.

(b) What is the maximum number of leaves in a tree with $n \geq 3$ vertices?

Answer:

(a) We give a direct proof. Consider the longest path $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}$ between two vertices $x = v_0$ and $y = v_k$ in the tree (here the length of a path is how many edges it uses, and if there are multiple longest paths then we just pick one of them). We claim that x and y must be leaves. Suppose the contrary that x is not a leaf, so it has degree at least two. This means x is adjacent to another vertex z different from v_1. Observe that z cannot appear in the path from x to y that we are considering, for otherwise there would be a cycle in the tree. Therefore, we can add the edge $\{z, x\}$ to our path to obtain a longer path in the tree, contradicting our earlier choice of the longest path. Thus, we conclude that x is a leaf. By the same argument, we conclude y is also a leaf.

The case when a tree has only two leaves is called the *path graph*, which is the graph on $V = \{1, 2, \ldots, n\}$ with edges $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$.

(b) We claim the maximum number of leaves is $n-1$. This is achieved when there is one vertex that is connected to all other vertices (this is called the *star graph*).

We now show that a tree on $n \geq 3$ vertices cannot have n leaves. Suppose the contrary that there is a tree on $n \geq 3$ vertices such that all its n vertices are leaves. Pick an arbitrary vertex x, and let y be its unique neighbor. Since x and y both have degree 1, the vertices x, y form a connected component separate from the rest of the tree, contradicting the fact that a tree is connected.

3. Edge-disjoint paths in hypercube

Prove that between any two distinct vertices x, y in the n-dimensional hypercube graph, there are at least n edge-disjoint paths from x to y (i.e., no two paths share an edge, though they may share vertices).

Answer: We use induction on $n \geq 1$. The base case $n = 1$ holds because in this case the graph only has two vertices $V = \{0, 1\}$, and there is 1 path connecting them. Assume the claim holds for the $(n-1)$-dimensional hypercube. Let $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ be distinct vertices in the n-dimensional hypercube; we want to show there are at least n edge-disjoint paths from x to y. To do that, we consider two cases:

1. Suppose $x_i = y_i$ for some index $i \in \{1, \ldots, n\}$. Without loss of generality (and for ease of explanation), we may assume $i = 1$, because the hypercube is symmetric with respect to the indices. Moreover, by interchanging the bits 0 and 1 if necessary, we may also assume $x_1 = y_1 = 0$. This means x and y both lie in the 0-subcube, where recall the 0-subcube (respectively, the 1-subcube) is the $(n-1)$-dimensional hypercube with vertices labeled $0z$ (respectively, $1z$) for $z \in \{0, 1\}^{n-1}$.

   Applying the inductive hypothesis, we know there are at least $n-1$ edge-disjoint paths from x to y, and moreover, these paths all lie within the 0-subcube. Clearly these $n-1$ paths will still be edge-disjoint in the original n-dimensional hypercube. We have an additional path from x to y that goes through the 1-subcube as follows: go from x to x', then from x' to y' following any path in the 1-subcube, and finally go from y' back to y. Here $x' = 1x_2 \ldots x_n$ and $y = 1y_2 \ldots y_n$ are the corresponding points of x and y in the 1-subcube. Since this last path does not use any edges in the 0-subcube, this path is edge-disjoint to the $n-1$ paths that we have found. Therefore, we conclude that there are at least n edge-disjoint paths from x to y.

2. Suppose $x_i \neq y_i$ for all $i \in \{1, \ldots, n\}$. This means x and y are two opposite vertices in the hypercube, and without loss of generality, we may assume $x = 00\ldots0$ and $y = 11\ldots1$. We explicitly exhibit n paths $P_1, \ldots, P_n$ from x to y, and we claim they are edge-disjoint.

   For $i \in \{1, \ldots, n\}$, the i-th path P_i is defined as follows: start from the vertex x (which is all zeros), flip the i-th bit to a 1, then keep flipping the bits one by one moving rightward from position $i+1$ to n, then from position 1 moving rightward to $i-1$. For example, the path P_1 is given by

$$000\ldots0 \rightarrow 100\ldots0 \rightarrow 110\ldots0 \rightarrow 111\ldots0 \rightarrow \cdots \rightarrow 111\ldots1$$

while the path P_2 is given by

$$000\ldots0 \rightarrow 010\ldots0 \rightarrow 011\ldots0 \rightarrow \cdots \rightarrow 011\ldots1 \rightarrow 111\ldots1$$

Note that the paths $P_1,\ldots,P_n$ don't share vertices other than $x = 00\ldots0$ and $y = 11\ldots1$, so in particular they must be edge-disjoint.

4. Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Answer:

Assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since v_1v_2 is not an edge, by property T it must be that v_1v and v_2v where $v \in \{v_5, v_6, v_7\}$ are edges. Similarly for v_3, v_4, v_3v and v_4v where $v \in v_5, v_6, v_7$ are edges. So now v_1, v_2, v_3 on one side and v_5, v_6, v_7 on the other form an instance of $K_{3,3}$. Contradiction.

5. Graph Coloring

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

Answer: The natural way to try to prove this theorem is to use induction on k. Unfortunately, this approach leads to disaster. It is not that it is impossible, just that it is extremely painful and would ruin your week if you tried it on an exam. When you encounter such a disaster when using induction on graphs, it is usually best to change what you are inducting on. In graphs, typical good choices for the induction parameter are n, the number of nodes, or e, the number of edges.

We use induction on the number of vertices in the graph, which we denote by n. Let $P(n)$ be the proposition that an n-vertex graph with maximum degree at most k is $(k+1)$-colorable.

Base Case $n = 1$: A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

Inductive Step: Now assume that $P(n)$ is true, and let G be an $(n+1)$-vertex graph with maximum degree at most k. Remove a vertex v (and all edges incident to it), leaving an n-vertex subgraph,H. The maximum degree of H is at most k, and so H is $(k+1)$-colorable by our assumption $P(n)$. Now add back vertex v. We can assign v a color (from the set of $k+1$ colors) that is different from all its adjacent vertices, since there are at most k vertices adjacent to v and so at least one of the $k+1$ colors is still available. Therefore, G is $(k+1)$-colorable. This completes the inductive step, and the theorem follows by induction.

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