CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand Discussion 8A Solutions

1. **Sock etiquette**

In your second week of Charm School you learn that you should only wear matching pair of socks. In each pair, both socks must be of the same color and pattern. But all of them are in one big basket and now you have to take a pair out. Let's say you own n pairs of socks which are all perfectly distinguishable (no two pairs have the same color and pattern). You are now randomly picking one sock after the other without looking at which one you pick.

(a) How many distinct subsets of k socks are there?

Answer: You could have interpreted the question to mean either that left and right socks are distinguishable from each other, or they are indistinguishable from each other.

For distinguishable left and right socks: We are picking k socks from $2n$ distinguishable socks, so there are $\binom{2n}{k} = \frac{2n!}{(2n-k)!k!}$ distinct subsets of k socks.

For indistinguishable left and right socks: This is a bit trickier. One way is to say that, for values $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$, first choose i pairs and take both socks from these pairs, and then choose $k-2i$ out of the remaining $n-i$ pairs and take one sock from these pairs. This gives the answer

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n}{i} \binom{n-i}{k-2i}.$$

You could have also told a different story and come up with an equation that looks different, but still gives you the same answer. For example, for $i \in \{0, \ldots, \lfloor k/2 \rfloor\}$, you take $k-i$ distinct socks (no two from the same pair), and then choose i out of those $k-i$ and add their matching socks to the subset. This gives the answer

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n}{k-i} \binom{k-i}{i},$$

which is the same as the previous formula.

(b) How many distinct subsets of k socks which do not contain a pair are there?

Answer:

When $k > n$, there are exactly 0 subsets of k distinct socks by the pigeonhole principle. Let's consider the case of interest when $k \leq n$.

For distinguishable left and right socks: One can construct a subset of $k \leq n$ socks which does not contain a pair by the following iterative process. Begin by picking any sock. While the number of picked socks is less than k, pick a sock belonging to a pair which has not yet appeared. In this process, we have $2n$ choices for the first sock, then $2(n-1)$ choices for the second one (as we can not pick the first sock again, nor pick the sock matching the first one), then $2(n-2)$ for the second sock, etc. Since we are counting subsets and the ordering does not matter, we divide everything by the number of ways to permute these k distinct socks, namely $k!$. Hence, there are

$$\begin{aligned}\frac{1}{k!} \times (2n \times 2(n-1) \times 2(n-2) \times \cdots \times 2(n-k+1)) &= \frac{1}{k!} \times \frac{2^k n!}{(n-k)!} \\ &= 2^k \binom{n}{k}\end{aligned}$$

such subsets.

For indistinguishable left and right socks: In this case, the answer is just $\binom{n}{k}$, since we can just take 1 sock from each pair and count the ways to make subsets of size k from this set.

(c) What is the probability of forming at least one pair when picking k socks out of the basket?

Answer: For distinguishable left and right socks:

We have:

$$\begin{aligned}\mathbf{P}(\text{there exists one pair of socks in set of } k) &= 1 - \mathbf{P}(\text{there is no pair of socks in the set of } k) \\ &= 1 - \frac{|\{k \text{ (distinct) all unpaired socks}\}|}{|\{k \text{ (distinct) socks}\}|} \\ &= 1 - \frac{2^k \binom{n}{k}}{\binom{2n}{k}} \\ &= 1 - \frac{2^k n!(2n-k)!}{(n-k)!(2n)!}\end{aligned}$$

You will notice that the $k!$ terms simplify, so we could have counted the ordered versions of questions a) and b) instead and obtained the same result.

For indistinguishable left and right socks: In this case, we can't use a counting argument, since different combinations are no longer equally likely, e.g., the probability you choose 1 of sock type 1 and 1 of sock type 2 is not the same as the probability that you choose 2 of sock type 1. We can, however, still use an independence argument, just like we did for collisions of balls in bins (recall Discussion 11M). In this case, let a "collision" mean that we pick two socks from the same pair. We will pick socks one at a time.

The probability of having no collisions when we pick the first sock is 1. For the second sock, the probability of no collisions is $\frac{2n-2}{2n-1}$, since we have $2n-1$ socks left to choose from and $(2n-1)-1$ of them won't result in a collision. For the third sock we pick, the probability of no collisions is $\frac{2n-4}{2n-2}$, since we have $2n-2$ socks left to choose from and $(2n-2)-2$ of them won't result in a collision (we can't pick the two that we've already picked). Continuing this pattern, we can see that the probability of picking k socks with no collisions, assuming $k \leq 2n$, is

$$\prod_{i=0}^{k-1} \frac{2n-2i}{2n-i} = 2^k \prod_{i=0}^{k-1} \frac{n-i}{2n-i} = 2^k \frac{n!/(n-k)!}{(2n)!/(2n-k)!} = \frac{2^k n!(2n-k)!}{(n-k)!(2n)!},$$

and therefore

$$\begin{aligned}\mathbf{P}(\text{there exists one pair of socks in set of } k) &= 1 - \mathbf{P}(\text{there is no pair of socks in the set of } k)\\ &= 1 - \frac{2^k n!(2n-k)!}{(n-k)!(2n)!},\end{aligned}$$

which is the same answer we got if we considered left and right socks to be distinguishable. It is important to recognize why, at an intuitive level, the probability is the same for both cases. We can pretend that someone secretly marks the right and left socks differently. But the person picking socks randomly can't see the marks and doesn't need to in order to pick a sock uniformly. A pair is a pair. So the probability must be the same.

(d) Now, in a different experiment, suppose there is exactly one sock of each pair in the basket (so there are n socks in the basket) and we sample (with replacement) k socks from the basket. What is the probability that we pick the same sock at least twice in the course of the experiment?

Answer: This is basically the birthday problem with n days and k people. The number of ways to sample k socks with replacement is n^k. The number of ways to sample k socks with replacement without repetition is $n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$. Hence, the probability that we sample the same sock at least twice in the course of the experiment is:

$$1 - \frac{n!}{(n-k)!n^k}.$$

2. **Company Selection**

Company A produces a particular device consisting of 10 components. Company A can either buy all the components from Company S or Company T, and then uses them to produce the devices without testing every individual component. After that, each device will be tested before leaving the factory. The device works only if every component works properly. Each working device can be sold for x dollars, but each non-working device must be thrown away. Products from Company S have a failure probability of $q = 0.01$ while Company T has a failure probability of $q/2$. However, every component from Company S costs \$10 while it costs \$30 from Company T. Should Company A build the device with components from Company S or Company T in order to maximize its expected profit per device? (Your answer will depend on x.)

Answer:

Let W denote the event that a device works. Let R be the random variable denoting the profit.

$$\mathbf{E}[R] = P(W)\mathbf{E}[R|W] + P(W^C)\mathbf{E}[R|W^C].$$

Let's first consider the case when we use products from Company S. In this case, a device works with probability $P[W] = (1-q)^{10}$. The profit made on a working device is $x - 100$ dollars while a nonworking device has a profit of -100 dollars. That is, $\mathbf{E}[R|W] = x - 100$ and $\mathbf{E}[R|W^C] = -100$. Using R_S to denote the profit using components from Company S, the expected profit is:

$$\mathbf{E}[R_S] = (1-q)^{10}(x-100) + (1-(1-q)^{10})(-100) = (1-q)^{10}x - 100 = (0.99)^{10}x - 100.$$

If we use products from Company T. The device works with probability $P[W] = (1-q/2)^{10}$. The profit per working device is $\mathbf{E}[R|W] = x - 300$ dollars while the profit for a nonworking device is $\mathbf{E}[R|W^C] = -300$ dollars. The expected profit is:

$$\mathbf{E}[R_T] = (1-q/2)^{10}(x-300) + (1-(1-q/2)^{10})(-300) = (1-q/2)^{10}x - 300 = (0.995)^{10}x - 300.$$

To determine which Company should we use, we answerve $\mathbf{E}[R_T] \geq \mathbf{E}[R_S]$, yielding $x \geq 200/[(0.995)^{10} - (0.99)^{10}] = 4280.1$. So for $x < \$4280.1$ using products from Company S results in greater profit, while for $x > \$4280.1$ more profit will be generated by using products from Company T.

3. **Dinner Time**

Now let's move on to the actual dinner. Each person has all sorts of plates, flatwares, and glasses in front of them, as shown in figure 1.[1] The basic rule is to start using utensils furthest from your plate and end with the closest ones. Table 1 lists the menu and the corresponding utensils.

Menu	Plates	Flatwares	Glasses
Water	-	-	O
Red wine	-	-	P
White wine	-	-	Q
Bread	K	L	-
Soup	-	J	-
Salad	E	B	-
Fish	F	C, I	-
Meat	G	D, H	-
Dessert	-	M, N	-

Table 1: Courses and utensils

Figure 1: Formal dinner setting

[1]Source: http://damoneroberts.tumblr.com/post/51078389219/home-tip-of-the-day-proper-place-settin

(a) Ron is confused what utensils to use (*'Wait, I think I'm at the wrong Charm School..'*). Fortunately, he can wait for his server to select the right plates and glasses. He just needs to pick flatwares. All he sees are, 4 forks (B, C, D, and N), 3 knives (H, I, and L), and 2 spoons (J and M). So, for each course served, he mimics what other people are using. For example, if other people are using a fork and a knife, he picks one fork and one knife. (He can't tell the difference between each fork, but can separate forks from knives and spoons just fine.) What is the probability he uses all utensils correctly? Each utensil is collected after each course and can't be used twice.

Answer: The total number of ways to permute the forks, knives, and spoons is $4! \cdot 3! \cdot 2!$. Since there is only 1 right way, the probability is $\frac{1}{4!3!2!} = \frac{1}{288}$. □ **Answer:**

We can think of this as Ron mentally making 3 separate orderings for forks, knives, and spoons. For each plate he nds out what kind(s) of flatware to use, then pick them according to the orders in his 3 lists. For example, let his lists be, forks: D, C, N, B, knives: I, H, L, and spoons: M, J, then he picks I for bread, M for soup, D for salad, C and H for sh, N and L for meat, and B and J for dessert. See Table 2 for illustration.

Menu	Forks	Knives	Spoons
Bread	-	I	-
Soup	-	-	M
Salad	D	-	-
Fish	C	H	-
Meat	N	L	-
Dessert	B	-	J

Table 2: How Ron can use flatwares according to the example orderings (in columns).

There are 4! ways to order the forks, 3! ways to order the knives, and 2! ways to order the spoons. So there are 4!3!2! ways to make these 3 lists. Since there is only one correct way to pick all the flatwares, the probability he uses all utensils correctly is $\frac{1}{4!3!2!} = \frac{1}{288}$. □

(b) Luna just doesn't care. For each course she just picks one or two random flatwares so that all of them are used at the end, and forces the server to serve on one random plate. For each drink she picks a random glass. What is the probability she used at least two things wrong? (If a utensil isn't used in the course it is matched with, then it is used wrongly.)

Answer: Again, we can view choosing utensils for each meal as just virtually arranging and using them in order. There are 4! and 3! ways to arrange plates and glasses. For flatwares, it is trickier.

Luna can use 1 or 2 per dish. Since there are 9 of them and 6 dishes to use with, they must be partitioned into 3 groups of one and 3 groups of two. First, we permute all flatwares and make Luna use the first three individually, and the last 6 in pairs (or a partition of 111222). This gives $\frac{9!}{2!2!2!}$ ways to arrange them since we permute 9 flatware items but the order does not matter for those grouped in pairs. Now, the partition 111222 can also be permuted (111222, 121122, 121212, ... so on), with a total number of $\frac{6!}{3!3!}$ or $\binom{6}{3}$ ways. Therefore, the total number of ways Luna can choose for all flatwares is

$$\frac{9!6!}{2!2!2!3!3!} \times 4! \times 3! = \frac{9!6!}{2!} = 1.306 \times 10^8.$$

The probability of using at least two things wrong = 1 $-$ probability of using at most one thing wrong. There is only one way to use everything correctly. The only place Luna can use just one thing wrong is with the flatwares where one of the paired flatwares is used with a flatware that is supposed to be used alone, otherwise at least two things must be wrong (because the other utensil whose place has been taken must also be wrong). For a pair of flatwares, there are 2 ways to pick just one of them, and 3 flatwares that it can be paired with (those that are supposed to be used alone). There are 3 pairs of flatwares, so the total number of ways to use one thing wrong = $2 \times 3 \times 3 = 18$.

Therefore, the probability that Luna used at least two things wrong = $1 - \dfrac{1+18}{9!6!/2} \approx 1$. □

(c) (*Optional*) What is the probability Hermione used all correct plates, flatwares, and glasses?

Answer: 1, duh. □